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Convergence Analysis for the Chebyshev Collocation Methods to Volterra Integral Equations with a Weakly Singular Kernel

Xiong Liu[1,3] and Yanping Chen[2,*]

[1] *Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Department of Mathematics, Xiangtan University, Xiangtan, Hunan 411105, China*
[2] *School of Mathematical Sciences, South China Normal University, Guangzhou, Guangdong 510631, China*
[3] *School of Mathematics and Statistics, Lingnan Normal University, Zhanjiang, Guangdong 524048, China*

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Abstract. In this paper, a Chebyshev-collocation spectral method is developed for Volterra integral equations (VIEs) of second kind with weakly singular kernel. We first change the equation into an equivalent VIE so that the solution of the new equation possesses better regularity. The integral term in the resulting VIE is approximated by Gauss quadrature formulas using the Chebyshev collocation points. The convergence analysis of this method is based on the Lebesgue constant for the Lagrange interpolation polynomials, approximation theory for orthogonal polynomials, and the operator theory. The spectral rate of convergence for the proposed method is established in the L^{∞}-norm and weighted L^2-norm. Numerical results are presented to demonstrate the effectiveness of the proposed method.

AMS subject classifications: 65M10, 78A48

Key words: Chebyshev collocation method, Volterra integral equations, spectral rate of convergence, Hölder continuity.

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1 Introduction

Integro-differential equations provide an important tool for modeling physical phenomena in various fields of science and engineering. This work is concerned with applying

*Corresponding author.
Emails: yanpingchen@scnu.edu.cn (Y. P. Chen), liuxiong980211@163.com (X. Liu)

Chebyshev spectral methods to solve Volterra integral equations (VIEs) of second kind with a weakly singular kernel

$$y(t)=g(t)+\int_0^t (t-s)^{-\frac{1}{2}}K(t,s)y(s)ds, \quad 0\le t\le T, \tag{1.1}$$

where the function $y(t)$ is the unknown function whose value is to be determined in the interval $0\le t\le T<\infty$. Here, $g(t)$ is a given smooth function and $K(t,s)$ is a given kernel, which is also assumed to be smooth.

For any positive integer m, if g and K have continuous derivatives of order m, then there exists a function $Z=Z(t,v)$ possessing continuous derivatives of order m, such that the solution of (1.1) can be written as $y(t)=Z(t,\sqrt{t})$, see, e.g., [3, 23, 24]. This implies that near $t=0$ the first derivative of the solution $y(t)$ behaves like $y'(t)\sim t^{-\frac{1}{2}}$. Several methods have been proposed to recover high order convergence properties for (1.1) using collocation type methods, see, e.g., [1, 2, 5, 8, 18, 19, 21, 22] and using multi-step method, see, e.g., [10, 25–27]. For spectral methods, the singular behavior of the exact solution makes the direct application of the spectral approach difficult. More precisely, for any positive integer m, we have $y^{(m)}(t)\sim t^{\frac{1}{2}-m}$, which indicates that $y\notin H^m_\omega(0,T)$, where H^m_ω is a standard Soblev space associated with a weight ω. To overcome this difficulty, we first apply the transformation

$$\tilde{y}(t)=t^{\frac{1}{2}}[y(t)-y(0)]=t^{\frac{1}{2}}[y(t)-g(0)] \tag{1.2}$$

to change (1.1) to the equation

$$\tilde{y}(t)=\tilde{g}(t)+t^{\frac{1}{2}}\int_0^t s^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}}K(t,s)\tilde{y}(s)ds, \quad 0\le t\le T, \tag{1.3}$$

where

$$\tilde{g}(t)=t^{\frac{1}{2}}[g(t)-g(0)]+t^{\frac{1}{2}}g(0)\int_0^t (t-s)^{-\frac{1}{2}}K(t,s)ds. \tag{1.4}$$

It is easy to see that the solution of (1.3) is a regular function

$$\tilde{y}(t)\in C^m([0,T]). \tag{1.5}$$

For the sake of applying the theory of orthogonal polynomials, we use the change of variable

$$t=\frac{T}{2}(1+x), \quad x=\frac{2}{T}t-1, \tag{1.6}$$

to rewrite the weakly singular problem (1.3) as follows

$$u(x)=f(x)+\left[\frac{T}{2}(1+x)\right]^{\frac{1}{2}}\int_0^{\frac{T}{2}(1+x)} s^{-\frac{1}{2}}\left(\frac{T}{2}(1+x)-s\right)^{-\frac{1}{2}}K\left(\frac{T}{2}(1+x),s\right)\tilde{y}(s)ds, \tag{1.7}$$