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An Extrapolation Cascadic Multigrid Method for Elliptic Problems on Reentrant Domains

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Abstract. This paper proposes an extrapolation cascadic multigrid (EXCMG) method to solve elliptic problems in domains with reentrant corners. On a class of λ-graded meshes, we derive some new extrapolation formulas to construct a high-order approximation to the finite element solution on the next finer mesh using the numerical solutions on two-level of grids (current and previous grids). Then, this high-order approximation is used as the initial guess to reduce computational cost of the conjugate gradient method. Recursive application of this idea results in the EXCMG method proposed in this paper. Finally, numerical results for a crack problem and an L-shaped problem are presented to verify the efficiency and effectiveness of the proposed EXCMG method.

AMS subject classifications: 65N55, 65N30

Key words: Richardson extrapolation, Cascadic multigrid, graded mesh, elliptic problems, corner singularity.

1 Introduction

It is well known that when an elliptic boundary value problem is solved by the finite element (FE) method on a quasi-uniform grid, the convergence rate is determined by the regularity of the solution [18]. Solutions of elliptic boundary value problems on domains with reentrant corners have singular behavior near the corners. This occurs even when

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the data of the underlying problem are very smooth. Such singular behavior significantly affects the accuracy of the FE method throughout the whole domain. For simplicity, we consider the Poisson equations with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is a given function in the $L^2(\Omega)$ and Ω is an open, bounded domain in $\mathbb{R}^2$ with at least one reentrant corner.

The unique solution of problem (1.1) belongs to $H^{1+\pi/\omega-\epsilon}$ for any $\epsilon > 0$ where ω is the maximum of the reentrant angles, and the standard continuous piecewise linear FE on a quasi-uniform grid yields $\mathcal{O}(h^{\pi/\omega-\epsilon})$ and $\mathcal{O}(h^{2\pi/\omega-\epsilon})$ accuracy in the H^1 and the L^2 norms, respectively. There are several approaches (such as conformal transformation methods, local mesh refinement and singular function methods) in the literature to overcome this difficulty (see [1–3, 7–12, 19] and references therein).

Locally refined grid for singular solutions was first studied by Babuska [1, 2]. Then, Schatz and Wahlbin [34] introduced a more general, locally refined grid to handle singularity. With this grid, optimal order convergence rate can be obtained. Huang and Lin [28, 29] proposed a radial shrinkage transformation method, which greatly simplifies the error analysis of FE approximations. The advantage of the local mesh refinement is that the knowledge of the exact form of the singular functions is not needed, so it becomes one of the most important methods to solve singular problems. However, when the singularity of solutions is intensive or a high accuracy is required, the number of grid points increases significantly. In these circumstance, classical iterative methods fail to be effective.

The multigrid (MG) method is regarded as one of the most effective methods for solving Poisson problems. In [27], Huang and Mu studied the extrapolation and MG algorithms for stress intensity factors on reentrant domains. Brenner [7–9] successfully applied the classical MG methods to compute the regular part of singular solutions on corner domains. Cai and Kim developed a new FE method based MG solvers by using singular functions for the Poisson equation on a polygonal domain with reentrant angles [10, 11]. However, classical MG methods have to cycle between coarse and fine grids to accelerate their rate of convergence. The cascadic multigrid (CMG) method proposed in 1996 by Deuflhard and Bornemann is a simpler multi-level method without coarse-grid correction [5]. Since then, many scholars have carried out extensive studies on the theoretical analysis and application of the method [6, 35, 36, 38, 39]. In 2008, we proposed an extrapolation cascadic multigrid method (EXCMG) for solving elliptic boundary value problems [14, 15]. The algorithm is based on the idea of CMG, while the linear interpolation on the coarse grid is modified as extrapolation and quadratic interpolation, which enable us to obtain a better initial guess for the iterative solution on the next finer grid. Then the conjugate gradient (CG) method is used to solve the resulting linear system with the good initial guess. After several years' development, the EXCMG algorithm has been successfully applied to non-smooth problems [22], parabolic problems [21], and