Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **8**, No. 6, pp. 911-931

DOI: 10.4208/aamm.2015.m1065
December 2016

Moving Finite Element Methods for a System of Semi-Linear Fractional Diffusion Equations

Jingtang Ma* and Zhiqiang Zhou

School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China

Received 26 April 2015; Accepted (in revised version) 17 November 2015

Abstract. This paper studies a system of semi-linear fractional diffusion equations which arise in competitive predator-prey models by replacing the second-order derivatives in the spatial variables with fractional derivatives of order less than two. Moving finite element methods are proposed to solve the system of fractional diffusion equations and the convergence rates of the methods are proved. Numerical examples are carried out to confirm the theoretical findings. Some applications in anomalous diffusive Lotka-Volterra and Michaelis-Menten-Holling predator-prey models are studied.

AMS subject classifications: 65M60, 65M12, 65M06, 35S10, 65R20

Key words: Finite element methods, fractional differential equations, predator-prey models.

1 Introduction

In this paper, we consider a system of semi-linear fractional diffusion equations of the following form

$$u_t - \mathscr{D}_1 \frac{\partial^{2-\beta_1} u}{\partial x^{2-\beta_1}} = f_1(u,v) + h_1(x,t), \tag{1.1a}$$

$$v_t - \mathscr{D}_2 \frac{\partial^{2-\beta_2} v}{\partial x^{2-\beta_2}} = f_2(u,v) + h_2(x,t), \tag{1.1b}$$

$$u(x,0) = \varphi(x), \qquad u(a,t) = u(b,t) = 0, \tag{1.1c}$$

$$v(x,0) = \psi(x), \qquad v(a,t) = v(b,t) = 0, \tag{1.1d}$$

for $(x,t) \in \Omega \times \mathbb{T}$ with $\Omega = (a,b)$, $\mathbb{T} = (0,T)$, where the functions f_i and h_i, positive constants $\mathscr{D}_i$ for $i=1,2$, and φ, ψ are given, and assume that f_i $(i=1,2)$ satisfy the following mixed local Lipschitz conditions

$$|f_i(u_1,u_2) - f_i(v_1,v_2)| \le L(|u_1 - v_1| + |u_2 - v_2|), \quad i = 1,2, \tag{1.2}$$

*Corresponding author.
Email: mjt@swufe.edu.cn (J. T. Ma), zqzhou@2014.swufe.edu.cn (Z. Q. Zhou)

for all u_1, u_2, v_1, $v_2 \in \Theta \subset \mathbb{R}$, where L is positive constant, and the space fractional derivatives are defined by

$$\frac{\partial^{2-\beta_i} u(x,t)}{\partial x^{2-\beta_i}} = D[p_a D_x^{-\beta_i} + q_x D_b^{-\beta_i}] D u(x,t), \tag{1.3}$$

where D denotes a single partial derivative, ${}_aD_x^{-\beta_i}$ and ${}_xD_b^{-\beta_i}$ represent left and right Riemann-Liouville fractional integral operators, $0<\beta_i<1$ $(i=1,2)$, p and q are two constants satisfying that $0\le p,q\le 1$, $p+q=1$.

The above model has many applications in population growth modeling (see e.g., [1, 2, 4, 5, 26]). Baeumer et al. [1, 2] studied the anomalous diffusion (fractional diffusion) population growth model for single specie. For competitive predator-prey models, the standard second-order diffusion models have been well studied (see e.g., [4,5,26]); however the studies on anomalous diffusion models are not many in the literature (we are only aware that Yu, Deng and Wu [47] studied the finite difference methods for the competitive predator-prey models with anomalous diffusion).

More specifically, the anomalous diffusion (fractional diffusion) predator-prey models studied in this paper include Lotka-Volterra and Michaelis-Menten-Holling types. Let u and v denote the population densities of prey and predator, respectively, $\mathscr{D}_i$ $(i=1,2)$ the coefficients of dispersion. If $h_i(x,t)\equiv 0$ for $i=1,2$, then the system is closed, i.e., u and v will develop freely, without influence from outside.

The competitive Lotka-Volterra models are described as follows. Let

$$f_1(u,v) = r_1 u\Big(1 - \frac{a_{11}u + a_{12}v}{K_1}\Big), \tag{1.4a}$$

$$f_2(u,v) = r_2 v\Big(1 - \frac{a_{22}v + a_{21}u}{K_2}\Big), \tag{1.4b}$$

where constants r_1, r_2 are inherent per-capita growth rates, constants K_1, K_2 are the carrying capacities, constants constants a_{12}, a_{21} represent the effect of the two species on each other, constants a_{11}, a_{22} are self-interacting factors for the two species. Then system (1.1a)-(1.1b) with (1.4a) and (1.4b) characterizes the well-known competitive Lotka-Volterra models (see e.g., [4]).

The Michaelis-Menten-Holling predator-prey model is a kind of ratio-dependent type predator-prey models. It is characterized by system (1.1a)-(1.1b) with the following Michaelis-Menten type functional response

$$f_1(u,v) = ru\Big(1 - \frac{u}{K} - \frac{d_1 v}{\kappa v + u}\Big), \tag{1.5a}$$

$$f_2(u,v) = v\Big(-Q(v) + \frac{d_2 u}{\kappa v + u}\Big), \tag{1.5b}$$

where d_1, d_2, κ, K, and r are positive constants. Here, $Q(v)$ denotes a mortality function of predator, and r and K the prey growth rate with intrinsic growth rate and the carrying capacity in the absence of predation, respectively, while d_1, d_2, and κ are model-dependent