Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **7**, No. 6, pp. 818-826

DOI: 10.4208/aamm.2014.m617
December 2015

Transient Diffusion in Triangular Cylinders

C. Y. Wang*

Departments of Mathematics and Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA

Received 13 May 2014; Accepted (in revised version) 12 November 2014

Abstract. A heated triangular cylinder is suddenly cooled in a constant temperature bath. The transient heat conduction problem is transformed to the Helmholtz equation related to the vibration of membranes. Using the membrane analogy, exact analytic solutions for the transient heat conduction problem for three triangular cross sections are found.

AMS subject classifications: 80A20

Key words: Diffusion, heat conduction, transient, triangular.

1 Introduction

Heat or mass diffusion is important in many diverse applications. One basic problem is the cooling of a heated solid which is suddenly introduced into a bath of lower temperature (quenching). The seminal work of Carslaw and Jaeger [1] discussed analytic solutions for the cooling of cylindrical solids whose cross sections are rectangular, circular, annular, or sectorial. Note that these geometries have boundaries that can be described by separable coordinates, such that decoupled ordinary differential equations result. For other cross sections numerical integrations are usually needed.

In this note we shall study the cooling of three special triangular cylinders. Since triangular boundaries cannot be described by separable coordinates, analytic methods such as separation of variables or Laplace transform described in [1] are difficult to apply. We shall show the cooling of these triangular cross sections is related to the eigenfunctions for the Helmholtz equation. Thus an analogy exists.

*Corresponding author.
Email: cywang@mth.msu.edu (C. Y. Wang)

2 Formulation

The transient heat transfer in a solid cylinder is governed by the diffusion equation

$$\nabla^2 T' = k\frac{\partial T'}{\partial t'}. \tag{2.1}$$

Here, ∇^2 is the Laplace operator in two dimensions, T' is the temperature, k is the thermal diffusivity, and t' is the time. Let the cross section have a (convenient) characteristic length L, original temperature ${T_0}'$ and ambient temperature ${T_a}'$. Normalize all lengths by L, the time and the temperature by

$$t = \frac{t'}{kL^2}, \quad T = \frac{T' - {T_a}'}{{T_0}' - {T_a}'}. \tag{2.2}$$

Eq. (2.1) becomes

$$\nabla^2 T = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) T = \frac{\partial T}{\partial t}, \tag{2.3}$$

where x, y are Cartesian coordinates describing the cross section. The boundary condition is

$$T = 0 \quad \text{on } S, \tag{2.4}$$

where S is the boundary of the cross section, and initially inside the solid

$$T = 1 \quad \text{at } t = 0. \tag{2.5}$$

Using separation of space and time variables on Eq. (2.3), the solution can be expressed in series form

$$T = \sum_i A_i \varphi_i(x,y) e^{-\lambda_i^2 t}, \tag{2.6}$$

where A_i are constant coefficients to be determined. Eq. (2.3) then yields the Helmholtz equation

$$\nabla^2 \varphi_i + \lambda_i^2 \varphi_i = 0. \tag{2.7}$$

Here λ_i^2 are the eigenvalues and $\varphi_i(x,y)$ are the corresponding two-dimensional eigenfunctions. Completeness can be shown. From Eq. (2.7) construct $(\varphi_j \nabla^2 \varphi_i - \varphi_i \nabla^2 \varphi_j)$ and integrate over the cross section Ω

$$\iint_\Omega (\varphi_j \nabla^2 \varphi_i - \varphi_i \nabla^2 \varphi_j) d\Omega = (\lambda_j^2 - \lambda_i^2) \iint_\Omega \varphi_i \varphi_j d\Omega. \tag{2.8}$$