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Discrete Maximum Principle Based on Repair Technique for Finite Element Scheme of Anisotropic Diffusion Problems

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Abstract. In this paper, we construct a global repair technique for the finite element scheme of anisotropic diffusion equations to enforce the repaired solutions satisfying the discrete maximum principle. It is an extension of the existing local repair technique. Both of the repair techniques preserve the total energy and are easy to be implemented. The numerical experiments show that these repair techniques do not destroy the accuracy of the finite element scheme, and the computational cost of the global repair technique is cheaper than the local repair technique when the diffusion tensors are highly anisotropic.

AMS subject classifications: 65M12, 65M60

Key words: Discrete maximum principle, finite element scheme, repair technique, anisotropic diffusion problems.

1 Introduction

Anisotropic diffusion equations appear in many physical models describing subsurface flows, heat conduction in structured materials, biological systems and plasma physics. A good diffusion scheme should be not only stable and convergent, but also possesses the mathmatical property of the physical system, such as the discrete maximum principle (DMP). The discrete maximum principle means, if the source term is non-positive, then

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the solution attains its maximum on the boundary. In the context of the anisotropic thermal conduction, a discrete scheme without satisfying the discrete maximum principle can lead to the violation of the entropy constraints of the second law of thermodynamics, causing heat to flow from regions of lower temperature to higher temperature. In the region of high temperature variations, this can cause the temperature to become negative. Therefore, the discrete maximum principle is an essential requirement in such diffusion processes to avoid the occurrence of unphysical phenomena.

As we know, it is very difficult to make the solution satisfy the discrete maximum principle. The classical finite volume and finite element schemes fail to satisfy the discrete maximum principle for strong anisotropic diffusion tensors or on highly distorted meshes [16–18]. The multi-point flux approximation (MPFA) method [10–12] and the mimetic finite difference (MFD) method [13,14] are second-order accurate on shape-regular meshes, but when the diffusion tensors are anisotropic or the meshes are highly distorted, these methods do not satisfy the discrete maximum principle. The diamond scheme [15], which is the so-called nine-point scheme on arbitrary quadrilateral meshes, is popular in solving diffusion equations. This method is a linear scheme and can be used on various distorted meshes for both smooth and non-smooth highly anisotropic solutions. However, this scheme is only positive-preserving and does not satisfy the discrete maximum principle. In the finite element schemes, the discrete maximum principle is satisfied by imposing severe restrictions on the choice of basis functions and on the geometric properties of the mesh. For a triangulation of acute or non-obtuse type (all angles smaller than or equal to $\pi/2$) the piecewise-linear finite element approximation of the Poisson equation satisfies DMP [4]. In the case of bilinear finite elements, it is sufficient to require that all quadrilaterals be of non-narrow type (aspect ratios smaller than or equal to $\sqrt{2}$). Recently, a nonlinear Galerkin finite element method [5] is proposed for isotropic Laplace equation on distorted meshes. Based on the repair technique and constrained optimization, the methods addressed in [2, 3, 7] enforce the linear finite element solution and the mixed element solution satisfying the DMP. Since the quadratic optimization is used, it is very expensive to solve the problem as the number of unknowns is increased. A constrained finite element method is described in [1], which can solve the problems with smooth coefficients very well, but is not satisfactory for the discontinuous anisotropic diffusion problems.

Compared with the existing methods, the repair technique is cheap and effective. In addition, it preserves the same total energy and accuracy as the original discrete scheme. The authors in [7, 19–21] have proposed a local repair technique to enforce the linear finite element solutions satisfying DMP. This technique repairs the out-of-bounds values one by one. Suppose a value below its minimum, we fix the value to the minimum, and the needed energy is taken from the neighborhood proportionally. If there isn't enough energy, we extend the neighborhood until enough energy is found. Then, the next value is checked and repaired if necessary. The procedure is similar to repair over-bounds values. Recently, a new repair technique [8], which is called the global repair technique, is addressed on finite volume diamond scheme for diffusion equations. In the present