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A New Finite Volume Element Formulation for the Non-Stationary Navier-Stokes Equations

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Abstract. A semi-discrete scheme about time for the non-stationary Navier-Stokes equations is presented firstly, then a new fully discrete finite volume element (FVE) formulation based on macroelement is directly established from the semi-discrete scheme about time. And the error estimates for the fully discrete FVE solutions are derived by means of the technique of the standard finite element method. It is shown by numerical experiments that the numerical results are consistent with theoretical conclusions. Moreover, it is shown that the FVE method is feasible and efficient for finding the numerical solutions of the non-stationary Navier-Stokes equations and it is one of the most effective numerical methods among the FVE formulation, the finite element formulation, and the finite difference scheme.

AMS subject classifications: 65N30, 65M30, 76M10

Key words: Non-stationary Navier-Stokes equations, finite volumes element method, error estimate, numerical simulations.

1 Introduction

Let $\Omega\subset\mathbb{R}^2$ be a bounded and connected polygonal domain. We consider the following incompressible non-stationary Navier-Stokes equations.

Problem 1.1. Find $\boldsymbol{u}=(u_1,u_2)$ and p such that, for $T>0$,

$$\begin{cases} \boldsymbol{u}_t-\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p=\boldsymbol{f}, & (x,y,t)\in\Omega\times(0,T),\\ \nabla\cdot\boldsymbol{u}=0, & (x,y,t)\in\Omega\times(0,T),\\ \boldsymbol{u}(x,y,t)=\boldsymbol{\varphi}(x,y,t), & (x,y,t)\in\partial\Omega\times(0,T],\\ \boldsymbol{u}(x,y,0)=\boldsymbol{u}_0(x,y), & (x,y)\in\Omega, \end{cases} \tag{1.1}$$

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where $\boldsymbol{u}=(u_1,u_2)$ represents the fluid velocity vector, p the pressure, T the total time, $\nu=1/Re$, Re the Reynolds number, $\boldsymbol{f}(x,y,t)$ the prescribed body force vector, $\boldsymbol{\varphi}(x,y,t)$ and $\boldsymbol{u}_0(x,y)$ are the boundary and initial values, respectively. For the sake of convenience, without loss of generality, we may as well suppose that $\boldsymbol{\varphi}(x,y,t)=\mathbf{0}$.

The system of the non-stationary Navier-Stokes equations is one of the important model system of equations in fluid dynamics. It has been successfully and extensively applied in many fields of practical engineering [1–3]. Due to its nonlinearity, there are no analytical solutions in general. One has to rely on numerical solutions. The finite volume element (FVE) method [4–6] is considered as one of the most effective numerical methods due to its following advantages. First, it preserves the integral invariants of conservation of mass as well as that of total energy. Second, it has higher accuracy and is more suitable for computations involving complicated boundary conditions than the finite difference (FD) method. Third, it has the same accuracy as the finite element (FE) method but is simpler and more convenient to apply than the FE method. It is also known as box method [7] or generalized difference method [8, 9]. Although it has been used to solve various types of partial differential equations, it focused on stationary partial differential equations and linear equations, for example elliptic problems, parabolic equations, Stokes equations, and viscoelastic problems, etc (see [4–17]).

Although some FVE methods for nonlinear Navier-Stokes equations have been provided, they are mainly based on stabilized and penalty FVE methods (see [18–21]). Even if the stabilized and penalty FVE methods for Navier-Stokes equations can enhance the stability of the numerical solutions and their theoretical analyses (e.g., stability and convergence) are conveniently achieved, the condition number of the coefficient matrices in their discrete systems would greatly increase. What's more, their numerical solutions would distort and diverge their accuracy solutions (in fact, the penalty term is an artificial viscosity). Thus, the theoretical study for fully discrete FVE method without any stabilization and penalty for the non-stationary Navier-Stokes equations holds more generality and more technologies required than those in [18–21]. So it has important theoretical meaning and practical value to do the theoretical analysis about the stability and error estimates of the fully discrete FVE method without any stabilization and penalty for non-stationary Navier-Stokes equations. Especially, to the best of our knowledge, as so far, there are no relative results published to do directly the theoretical study for the fully discrete FVE method without any stabilization and penalty for non-stationary Navier-Stokes equations. Therefore, we will do these studies in this paper and provide the numerical experiments for illustrating the feasibility and efficiency of FVE method without any stabilization and penalty. It is also shown that the FVE method is more stabile than FE method and FD scheme by comparing their numerical solutions. Especially, we here directly derive a new fully discrete FVE formulation based on macroelement without any stabilization and penalty from the semi-discrete formulation with respect to time and do theoretical study which could avoid the semi-discrete FVE formulation about spatial variable and satisfy discrete Babuška-Brezzi (B-B) inequality (see [23, 24]).