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Compact Finite Difference Scheme for the Fourth-Order Fractional Subdiffusion System

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Abstract. In this paper, we study a high-order compact difference scheme for the fourth-order fractional subdiffusion system. We consider the situation in which the unknown function and its first-order derivative are given at the boundary. The scheme is shown to have high order convergence. Numerical examples are given to verify the theoretical results.

AMS subject classifications: 26A33, 35R11, 65M06, 65M12, 65M15

Key words: Fourth-order fractional subdiffusion equation, compact difference scheme, energy method, stability, convergence.

1 Introduction

Recently, it has been found that many physical phenomena and processes in engineering can be modeled by fractional differential equations (FDEs), see [1–6]. Therefore, there have been growing interests on the study of these equations. The difference between classical differential equations and FDEs are that the derivatives involved in the equation are of fractional order. Since the definitions of fractional derivatives are completely different from that of the classic derivatives, it becomes necessary to develop the study of FDEs in both theoretical and numerical aspects. This paper concentrates on numerical methods for solving FDEs.

In the past decade, many works have been done on the study of efficient methods for the numerical approximation of FDEs. Our interest lies in the finite difference method. Here, we give a brief review of some recent progress in this direction. In [7], Tadjern and Meesrschaert studied a Crank-Nicolson method with second-oder accuracy in time

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by a special extrapolation. The method was then generalized to two-dimensional problems by the alternating directions implicit method [8]. By the use of the energy method and the method of order reduction, Sun and Wu [9] studied the stability and convergence of a fully discrete scheme for a fractional diffusion-wave equation. Based on [9], a compact finite difference scheme was studied in [10]. Compact scheme with high order accuracy is our main concern. In [11], Cui gave a compact finite difference scheme with spatial accuracy of fourth order for fractional diffusion equations. A compact finite difference scheme for fractional subdiffusion equations was shown to converge with order $\mathcal{O}(\tau^{2-\alpha}+h^4)$ in [12], where τ is the temproal grid size and h is the spatial grid size. The technique used in [12] has also been applied to a generalized Cattaneo equation in [13]. FDEs with different types of boundary conditions are also of interest. In a very recent result, Ren et al. [14] established a fourth order compact scheme for the fractional subdiffusion equation with Neumann boundary conditions.

All the results mentioned above concern with equations having second-order space derivatives. However, in some practical applications, the problem must be modeled by fourth-order space derivatives. For example, when modeling the formation of grooves on a flat surface because of grain, fourth-order terms are required [15, 16]. FDEs with fourth-order space derivatives were studied in [17–19] on both bounded and unbounded domain. Homotopy perturbation method was employed in [20] to obtain the approximate solution of a generalized fourth-order fractional diffusion-wave equation. Very recently, a finite difference scheme for fourth-order FDE was studied in [21]. This result was further improved in [22], where a compact finite difference scheme for a fractional diffusion-wave equation was shown to converge with order $\mathcal{O}(\tau^{3-\alpha}+h^4)$, when the values of the unknown function u and u_{xx} are given at the boundary.

Inspired by the results of [21,22], in this paper, we consider, for $\alpha\in(0,1)$, the following fractional subdiffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\kappa^2\frac{\partial^4 u}{\partial x^4}=f(x,t),\quad 0\le x\le 1,\quad 0<t\le T, \tag{1.1}$$

subject to the initial condition

$$u(x,0)=\phi(x),\quad 0\le x\le 1, \tag{1.2}$$

and the boundary conditions

$$u(0,t)=g_L(t),\quad u(1,t)=g_R(t),\quad \frac{\partial u(0,t)}{\partial x}=\tilde{g}_L(t),\quad \frac{\partial u(1,t)}{\partial x}=\tilde{g}_R(t),\quad 0<t\le T, \tag{1.3}$$

where $\partial^\alpha u/\partial t^\alpha$ is the Caputo fractional derivative of u which is defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}$$

with $\Gamma(\cdot)$ being the gamma function.