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Conservative and Finite Volume Methods for the Convection-Dominated Pricing Problem

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Abstract. This work presents a comparison study of different numerical methods to solve Black-Scholes-type partial differential equations (PDE) in the convection-dominated case, i.e., for European options, if the ratio of the risk-free interest rate and the squared volatility-known in fluid dynamics as Péclet number-is high. For Asian options, additional similar problems arise when the "spatial" variable, the stock price, is close to zero.

Here we focus on three methods: the exponentially fitted scheme, a modification of Wang's finite volume method specially designed for the Black-Scholes equation, and the Kurganov-Tadmor scheme for a general convection-diffusion equation, that is applied for the first time to option pricing problems. Special emphasis is put in the Kurganov-Tadmor because its flexibility allows the simulation of a great variety of types of options and it exhibits quadratic convergence. For the reduction technique proposed by Wilmott, a put-call parity is presented based on the similarity reduction and the put-call parity expression for Asian options. Finally, we present experiments and comparisons with different (non)linear Black-Scholes PDEs.

AMS subject classifications: 65M10, 91B25

Key words: Black-Scholes equation, convection-dominated case, exponential fitting methods, fitted finite volume method, Kurganov-Tadmor scheme, minmod limiter.

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1 Introduction

An option is an instrument in which two parties agree to the possibility to exchange an asset, the underlying, at a predefined price and maturity. The payoff represents the

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profit-loss profile defined by the option for a given range of prices. There exist a great variety of options ranging from European options, to American, Asian, Barrier options, Binary options, etc., and many of these derivatives are valuated with the classical pricing formulae developed by Black, Scholes and Merton. The type of the option refers in many cases to the type of payoff profile of the option but for the European and American option, the type refers to its maturity: the maturity of a European option is fixed whereas the American option can be exercised at any time before the maturity, e.g., there exist Asian options of European and American type.

The partial differential equation (PDE) proposed by Black, Scholes and Merton is known as the Black-Scholes equation and is a special case of a more general convection-diffusion equation that also arises in other areas like fluid dynamics. Due to its convective term, the solution is a traveling wave transporting the initial data and due to the diffusive term the data is dissipated: a dissipating traveling wave. If the diffusion coefficient is small compared to to the transport coefficient then the solution behaves mainly like a traveling wave and the equation is said to be convection-dominated. For purely first order hyperbolic PDEs, it is known [9] that standard methods fail to obtain an acceptable approximation when discontinuities are present in the initial data and a similar issue is observed in the convection-diffusion equation in the convection-dominated case and discontinuous initial data. Some schemes like the Lax-Friedrichs or the upwind method were proposed to obtain satisfactory approximations for hyperbolic PDEs, but artificial diffusion is introduced by the method which leads to smeared solutions. In terms of the Black-Scholes equation, this behavior appears if the squared volatility is small in comparison with the risk-free rate.

When solving numerically the Black-Scholes PDE it is useful to reverse transform the time variable to use the payoff function (terminal condition) as the initial condition (IC) of the system. Albeit the payoff of an European option is non-smooth and the numerical solution for the price is acceptable, artificial oscillations appear near the strike price when the first numerical derivative with respect to the underlying price of this approximation is obtained. These oscillations are worst when higher derivatives are calculated. Having access to the first derivative of the option price is important to measure the sensitivity of the option to movements on the price of the underlying or other parameters like volatility. Higher derivatives of the option price also provide important information about the behavior of the option. These quantities are known in the financial literature as the *Greeks*. Due to the Greeks being relevant for the quantitative analysts, reliable numerical methods are required for the pricing of options which not only provide a good approximation for the price, but also for its derivatives.

In this paper finite difference methods (FDMs) for the Black-Scholes equation with a wide range of parameters, including the critical convection-dominated case are considered. Conservative methods, a special family of FDMs and also denoted as finite volume methods (FVMs), are presented as the method of choice to solve convection-dominated problems. Two conservative methods are studied: the Kurganov-Tadmor scheme [14], a high resolution method for a general convection-diffusion equation which exhibits