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A Priori Error Estimates of Crank-Nicolson Finite Volume Element Method for Parabolic Optimal Control Problems

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Abstract. In this paper, the Crank-Nicolson linear finite volume element method is applied to solve the distributed optimal control problems governed by a parabolic equation. The optimal convergent order $\mathcal{O}(h^2+k^2)$ is obtained for the numerical solution in a discrete L^2-norm. A numerical experiment is presented to test the theoretical result.

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Key words: Variational discretization, parabolic optimal control problems, finite volume element method, distributed control, Crank-Nicolson.

1 Introduction

The optimal control problems introduced in [13] are playing an increasingly important role in science and engineering. They have various applications in the operation of physical, social, and economic processes. Many numerical methods, such as finite element method, mixed finite element method, spectral method, streamline finite element method etc., have been applied to approximate the solutions of these problems (see, e.g., [3–8, 10, 14]).

In [16], to our best knowledge, the authors first use the finite volume element method to obtain the numerical solution for an optimal control problem associate with a parabolic

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equation by using *optimize-then-discretize* approach and the variational discretization technique (proposed in [12]). Also, the authors derive some error estimates for the semi-discrete solution and fully-discrete approximation. For the fully-discrete approximation, the convergent order is $\mathcal{O}(h^2+k)$ there. Here we develop the Crank-Nicolson linear finite volume element method for solving the parabolic optimal control problems and get the optimal order $\mathcal{O}(h^2+k^2)$.

In this paper, we consider the following optimal control problems: Find y, u such that

$$\min_{u\in U_{ad}} \frac{1}{2}\int_0^T \big(\|y(\tau,x)-y_d(\tau,x)\|^2_{L^2(\Omega)}+\alpha\|u(\tau,x)\|^2_{L^2(\Omega)}\big)d\tau, \tag{1.1a}$$

$$y_t(t,x)-\nabla\cdot(A\nabla y(t,x))=Bu(t,x)+f(t,x), \quad t\in J, \quad x\in\Omega, \tag{1.1b}$$

$$y(t,x)=0, \quad t\in J, \quad x\in\Gamma, \quad y(0,x)=y_0, \quad x\in\Omega, \tag{1.1c}$$

where

$$\nabla\cdot(A\nabla y)=\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big),$$

$\Omega\subset R^2$ is a bounded convex polygon domain and Γ is the boundary of Ω, α is a positive number, $f(t,\cdot),y_d(t,\cdot)\in L^2(\Omega)$ or $H^1(\Omega)$, $J=(0,T]$, $A=(a_{i,j})_{2\times 2}$ is a symmetric, smooth enough and uniformly positive definite matrix in Ω, $B:L^2(J;L^2(\Omega))\to L^2(J;L^2(\Omega))$ is a continuous linear operator, $y_0(x)=0$, $x\in\Gamma$, U_{ad} is a set defined by

$$U_{ad}=\big\{u:\ u\in L^2(J;L^2(\Omega)),\ a\le u(t,x)\le b,\ a.e. \text{ in } \Omega,\ t\in J,\ a,b\in \mathrm{R}\big\}.$$

A semi-discrete optimal system is carried out in [16] and the existence and uniqueness of the solution for the system is proved there. Here we use the Crank-Nicolson scheme to discretize the semi-discrete optimal system and obtain the optimal convergent order $\mathcal{O}(h^2+k^2)$.

The remainder of this paper is organized as follows. In Section 2, we present some notations. In Section 3, we present the Crank-Nicolson linear finite volume element method for the optimal control problems. In Section 4, we first show some lemmas and then analyze the error estimate between the exact solution and the Crank-Nicolson linear finite volume element approximation. And in Section 5, a numerical example is presented to test the theoretical results.

Throughout this paper, the constant C denotes different positive constant at each occurrence, which is independent of the mesh size h and the time step k.

2 Notations

We use the standard notations $W^{m,p}(\Omega)$ for Sobolev spaces and their associated norms $\|v\|_{m,p}$ (see, e.g., [1]) in this paper. To simplify the notations, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and drop the index $p=2$ and Ω whenever possible, i.e.,

$$\|u\|_{m,2,\Omega}=\|u\|_{m,2}=\|u\|_m, \quad \|u\|_0=\|u\|.$$