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The Method of Fundamental Solutions for Solving Exterior Axisymmetric Helmholtz Problems with High Wave-Number

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Dedicated to Graeme Fairweather on the occasion of his 70^{th} birthday.

Abstract. In this paper, we investigate the method of fundamental solutions (MFS) for solving exterior Helmholtz problems with high wave-number in axisymmetric domains. Since the coefficient matrix in the linear system resulting from the MFS approximation has a block circulant structure, it can be solved by the matrix decomposition algorithm and fast Fourier transform for the fast computation of large-scale problems and meanwhile saving computer memory space. Several numerical examples are provided to demonstrate its applicability and efficacy in two and three dimensional domains.

AMS subject classifications: 65M10, 78A48

Key words: Method of fundamental solutions, exterior Helmholtz problem, circulant matrix, fast Fourier transform, axisymmetric domain.

1 Introduction

The method of fundamental solutions (MFS) [4, 11, 13, 14, 23], known for its simplicity and accuracy, has been gaining in popularity in various areas of scientific computing. Like the boundary element method (BEM) [3], it is applicable when the fundamental solution of the governing equation is known in advance. Numerical solution of the MFS is approximated by a linear combination of the fundamental solutions in terms of singularities which are placed outside the domain of the problem under consideration. The singularities of fundamental solutions can be either free or fixed which will result in, respectively, a nonlinear least square problem or a linear system [11]. A review of some

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related work as well as varieties of applications and advantages over other methods can be found in [12, 14, 16, 25, 27, 32].

It is known that the MFS is particularly efficient and accurate for solving exterior Helmholtz problems since it is a truly boundary-type meshless method and the fundamental solution of the governing equations naturally satisfies the Sommerfeld radiation condition [2, 12, 14]. On the other hand, it is also known that the coefficient matrix generated by the MFS is often dense and ill-conditioned [6,8,33]. Direct solver for such a matrix requires $\mathcal{O}(N^3)$ operations and $\mathcal{O}(N^2)$ storages. As a result, the MFS is not feasible for solving exterior Helmholtz problems with high wave-number since massive collocation points are required. Furthermore, when the number of collocation points is large, the coefficient matrix becomes extremely ill-conditioned [7, 26]. In the past, the domain decomposition method (DDM), regularization techniques etc. [1,24,34] have been proposed to alleviate the conditioning and storage problems associated with the MFS formulation. In general, it is known that the structure of the coefficient matrix is closely related to the distribution of collocation points. If the collocation points and the solution domain are chosen in a particular fashion, the resulting matrix system has a certain structure. Recently, efficient algorithms have been developed for solving axisymmetric homogeneous differential equations in context of the MFS [10, 17–21, 28, 29, 31].

In this paper, our main goal is to develop an efficient numerical algorithm to solve high wave-number exterior Helmholtz problems in axisymmetric domain which has not been reported in literatures. The main problems rest on the high wave-number and Neunamm boundary condition, for the fact that massive collocation points should be used to high wave-number problems and Neumann boundary condition may result in some structural damage to the coefficient matrix. By the radial properties of the fundamental solution and radial symmetric of the solution domain, we show the circulant or block circulant features of the coefficient matrices for problems under pure Dirichlet or Neumann boundary condition. And we take advantage of the special features of the circulant matrix [9] to accelerate the solution procedure. The key idea behind this algorithm is the matrix decomposition algorithm which has been widely circulated in science and engineering. For more details, we refer readers to the References [18–20]. Overall, the proposed algorithm decomposes the give system of equations into a series of linear systems of lower rank. Similar to the traditional matrix decomposition method, the proposed algorithm makes extensive use of fast Fourier transform which results in additional saving in the computational time and memory storage.

The rest of the paper is organized as follows. In Section 2, we give a general formulation of the MFS to exterior Helmholtz problems. In Section 3, through the use of the circulant matrix and fast Fourier transform, we describe how the matrix decomposition method is being used for solving both two and three dimensional problems. The efficacy of the proposed technique for solving Helmholtz problems with high wave-number is demonstrated by several benchmark examples in Section 4. In Section 5, we conclude with some discussion for the direction of future research.