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Polynomial Particular Solutions for the Solutions of PDEs with Variables Coefficients

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Abstract. The closed-form particular solutions with polynomial basis functions for general partial differential equations (PDEs) with constant coefficients have been derived and applied for solving various kinds of problems in the context of the method of approximate particular solutions (MAPS). In this paper, we propose to extend the above-mentioned method to PDEs with variable coefficients by the substituting and adding-back technique. Since the linear system derived from the polynomial particular solutions is notoriously ill-conditioned, the multiple scale method is applied to alleviate this difficulty. To validate our proposed method, four numerical examples are considered and compared with those obtained by the MAPS using the radial basis functions.

AMS subject classifications: 65N80, 65N35

Key words: Polynomial particular solutions, variable coefficients, multiple scale methods, collocation approach, method of approximate particular solutions.

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1 Introduction

The particular solution has traditionally played an important role in science and engineering aspects [1–4]. The particular solution is known to all that it satisfies the governing equations in the infinite domains which do not have to satisfy boundary conditions. In 1967, Fox et al. [5] first proposed the particular solutions for solving the eigenvalue

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problems governed by the Laplace equation where the solution is approximated by the Bessel functions and sine functions. In general, the considered problem can be solved if we can obtain the homogeneous solutions and particular solutions. However, it remains a challenge that the particular solutions and homogeneous solutions are not always available. It is known that particular solutions are not unique which can be derived by numerous ways [3, 6, 7]. In the last few decades, various numerical methods have been proposed for the approximations of particular solutions. In recent years, in contrast to the traditional meshed based methods [8–11], the radial basis functions (RBFs) have witnessed a research boom due to their effectiveness and simplicity of implementations [12–14]. The RBFs have been used as an effective tool to obtain closed-form particular solutions [15, 16]. Previously, particular solutions are subtracted off which will lead to homogeneous differential equations which can then be solved by the boundary-type methods, such as the boundary element method, the method of fundamental solutions and the singular boundary method [17–19]. This is a two-stage approach which is well-known in literatures.

Chen et al. [2, 3] proposed the method of approximate particular solutions (MAPS) where particular solutions are used as the basis functions to satisfy the governing equations and boundary conditions simultaneously as an one-stage method. In the MAPS, we try to find a solution of the problem using the following form

$$u(x,y)=\sum_{j=1}^{\infty}\alpha_j u_p^j(x,y), \tag{1.1}$$

where $u_p^j(x,y)$ is the particular solution of the given differential operator for some basis functions. Due to the rapid development in this area, the MAPS in the context of RBFs has been applied for solving a large class of problems in science and engineering. Despite the success of the RBFs, there are still challenges such as the determination of the shape parameter of RBFs and the difficulty in deriving closed-form particular solutions for general problems. As a result, Chebyshev polynomial functions have been adopted as an alternative to alleviate some of these difficulties [20–25]. These approaches have been proven to be highly accurate. However, the solution procedure is quite tedious and the closed-form particular solutions are only available for some specific differential operators [7]. One of the main disadvantages of using Chebyshev polynomials as the basis functions is that the forcing term of the differential equation should be smoothly extendible to the exterior of the solution domain in the case of irregular domains. As such, collocation points can be selected to be at the specific Gauss-Lobatto points.

Recently, the closed-form particular solutions of the general linear differential operators have been successfully derived by using the standard polynomial basis functions [26]. By coupling with the MAPS, a large class of partial differential equations have been solved. In [27], the polynomial particular solutions are used to simulate the plate bending vibration problems. The Cauchy problems are successfully solved by the polynomial particular solutions [28]. However, until now, it can only be used to solve