Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 4, pp. 990-1011

DOI: 10.4208/aamm.2015.m1312
August 2017

Numerical Simulation for the Variable-Order Fractional Schrödinger Equation with the Quantum Riesz-Feller Derivative

N. H. Sweilam* and M. M. Abou Hasan

Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

Received 4 October 2015; Accepted (in revised version) 8 September 2016

Abstract. In this paper the space variable-order fractional Schrödinger equation (VOFSE) is studied numerically, where the variable-order fractional derivative is described here in the sense of the quantum Riesz-Feller definition. The proposed numerical method is the weighted average non-standard finite difference method (WANSFDM). Special attention is given to study the stability analysis and the convergence of the proposed method. Finally, two numerical examples are provided to show that this method is reliable and computationally efficient.

AMS subject classifications: 65M06, 65M12, 34A08

Key words: Variable-order Schrödinger equation, quantum Riesz-Feller variable-order definition, weighted average non-standard finite difference method, Jon von Neumann stability analysis.

1 Introduction

In recent years, many mathematical, physical, engineering and financial phenomena have been successfully described using the variable-order fractional calculus, see [1, 3, 6, 9, 10, 15, 29, 34, 47, 50, 51, 54]. Thus, many mathematicians and physicists have been interested in studying the properties of variable-order fractional calculus and finding active and accurate analytical and numerical methods for solving variable-order fractional differential equations. Starting from 1993, Samko et al. [47] who proposed this interesting extension of the classical fractional calculus. Where in the concept of variable-order fractional derivative the order of the operator is allowed to vary either as a function of the independent variable of differentiation (t) or as a function of some other (perhaps spatial) variable (x) or both. The analytic results on the existence and uniqueness of the solutions for a generalized fractional differential equations with variable-order operators

*Corresponding author.
Email: nsweilam@sci.cu.edu.eg (N. H. Sweilam)

have been discussed in [6, 50, 51]. The effect of differences between using constant and variable-order fractional derivatives has been discussed in [15].

It is well known that the Schrödinger equation plays a fundamental role in quantum mechanics. It describes how the quantum state of some physical system changes with time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger.

Laskin [24–27] constructed the space fractional Schrödinger equation in form (1.1) as a generalization of the classical Schrö dinger equation, obtained by replacing the second-order space derivative by a Riesz fractional derivative.

$$i\hbar\frac{\partial\Psi(r,t)}{\partial t}=C_\alpha(m)(-\Delta)^{\alpha/2}\Psi(r,t)+V(r,t)\Psi(r,t), \quad t\geq 0, \quad r\in\mathbb{R}, \tag{1.1}$$

for the wave function Ψ of a quantum particle with the mass m that moves in a potential field with the potential V. Where $\hbar=\frac{h}{2\pi}$ (h is the Plank constant), $C_\alpha(m)$ is a positive constant which equals $\frac{\hbar^2}{2m}$ for $\alpha=2$ and $(-\Delta)^{\alpha/2}$ was called the quantum Riesz fractional derivative of order α. In the mathematical literature, $(-\Delta)^{\alpha/2}$ is usually referred to as the fractional Laplacian. For $\alpha=2$, the quantum Riesz fractional derivative becomes the negative Laplace operator $-\Delta$ and Eq. (1.1) is reduced to the classical Schrödinger equation for a quantum particle with the mass m that moves in a potential field with the potential V.

Many papers (see, e.g., [2, 17, 18, 27, 49, 53] and the references cited therein), using several analytical and numerical methods have dealt with the space-fractional and space-time-fractional Schrödinger equations with some specific potential fields including zero potential (free particle), the δ-potential, the infinite potential well, the Coulomb potential, and a rectangular barrier. In 2013, Al-Saqabi et al. [7] solved the fractional Schrödinger equation with the quantum Riesz-Feller derivative for a particle that moves in a potential field in terms of the Fox H-function. Atangana et al. [1] solved the space variable-order fractional Schrödinger equation using Crank-Nicholson scheme, they used the Caputo variable-order differential operator.

The non-standard finite difference method (NSFDM) is proposed by Mickens [35, 36] for improving spatial discretizations, such that depending on the denominator function and the specific discretization this method be more accurate and more stable than standard method [4, 33], in addition this method can be easy to formulate [41]. The positive applications of the NSFDM can be found in the fields of physics, chemistry, engineering [12, 20, 45]. Especially, the most attractive applications are in mathematical biology and ecology [46, 48] such that the merit of the NSFDM has been shown prominently. In addition, the dynamic preserving properties of the NSFDM are also well performed in solving fractional-order system, such as the fractional-order neuron system [21], the fractional-order Rössler system [19], and the fractional Hodgkin-Huxley model [5]. Also, the weighted average finite difference method (WAFDM) can be explicit method (easy for coding) or implicit method (more stable) depending on the weight factor [14,28,43]. Here, we try to merge between these two methods to find better scheme for VOFSE such that