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Analysis of an Implicit Fully Discrete Local Discontinuous Galerkin Method for the Time-Fractional Kdv Equation

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Abstract. In this paper, we consider a fully discrete local discontinuous Galerkin (LDG) finite element method for a time-fractional Korteweg-de Vries (KdV) equation. The method is based on a finite difference scheme in time and local discontinuous Galerkin methods in space. We show that our scheme is unconditionally stable and convergent through analysis. Numerical examples are shown to illustrate the efficiency and accuracy of our scheme.

AMS subject classifications: 65M60, 35K55

Key words: Time-fractional partial differential equations, Kdv equation, local discontinuous Galerkin method, stability, error estimates.

1 Introduction

In this paper, we consider the following time-fractional Korteweg-de Vries (KdV) equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\varepsilon g(u)_x+\nu\frac{\partial^3 u(x,t)}{\partial x^3}=f(x,t),\quad (x,t)\in[a,b]\times[0,T], \tag{1.1a}$$

$$u(x,0)=u_0(x),\quad x\in[a,b], \tag{1.1b}$$

where $0<\alpha\leq 1$ is a parameter describing the order of the fractional time, $\varepsilon,\nu\geq 0$ are parameters. f, u_0 are given smooth functions. We do not pay attention to boundary condition in this paper; hence the solution is considered to be either periodic or compactly supported.

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The time fractional derivative in the Eq. (1.1), uses the Caputo fractional partial derivative of order α, defined as [9]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}, & \text{if } 0<\alpha<1,\\ \dfrac{\partial u(x,t)}{\partial t}, & \text{if } \alpha=1,\end{cases} \tag{1.2}$$

here $\Gamma(\cdot)$ is the Gamma function.

Factional differential equations are increasingly used to model problems in rheology and mechanical systems, control and robotics, and other areas of applications. In recent years interest of some scholars has been shown in research on the problems involving the fractional order partial differential equations (PDEs) [3, 11, 14, 15, 18, 21, 27–29, 31–33]. Machado et al. [12] introduced the recent history of fractional calculus, as for the detailed theory and applications of fractional integrals and derivatives, we can refer to [13,19] and the references therein. Due to their numerous applications in the areas of physics and engineering, solving such equations and numerical schemes for fractional differential equations have been stimulated.

There are only a few numerical works in the literature to solve the fractional KdV equation. Momani [8] extended Adomian decomposition method to derive explicit and numerical solutions of the space-time fractional KdV equation. Abdulaziz et al. [1] applied the homotopy analysis method (HAM) and its modification (MHAM) to solve the nonlinear time- and space-fractional modified Korteweg-de Vries (fmKdV), and obtained some approximate and exact analytical solutions of the fmKdV. In the present paper we develop an implicit fully discrete local discontinuous Galerkin (LDG) finite element method for solving time-fractional KdV equation. This development is based on the extensive work on DG for problems founded in classic calculus [5,7,17,20,22–26]. The proposed method is different from the traditional LDG method, which discretes a equation in spatial direction and couples an ordinary differential equation (ODE) solver, such as Runger-Kutta method. Our fully discrete scheme is based on a finite difference scheme in time and local discontinuous Galerkin methods in space. Stability is ensured by a careful choice of interface numerical fluxes. We prove that our scheme is unconditional stable and convergent. Although many studies for the classical KDV time-dependent problems and some profound results have been established, it seems that detailed studies of the fractional KDV equation are only beginning.

What remains of this paper is organized as follows. First we introduce some basic notations and mathematical preliminaries, then in Section 3 we discuss the fully discrete LDG scheme for the fractional KDV equation (1.1), and prove that the scheme is unconditionally stable, and the numerical solution is convergent. Numerical experiments to illustrate the accuracy and capability of the method are given in Section 4. Finally in Section 5 concluding remarks are provided.