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Modification of Multiple Knot B-Spline Wavelet for Solving (Partially) Dirichlet Boundary Value Problem

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Abstract. A construction of multiple knot B-spline wavelets has been given in [C. K. Chui and E. Quak, Wavelet on a bounded interval, In: D. Braess and L. L. Schumaker, editors. Numerical methods of approximation theory. Basel: Birkhauser Verlag; (1992), pp. 57–76]. In this work, we first modify these wavelets to solve the elliptic (partially) Dirichlet boundary value problems by Galerkin and Petrov Galerkin methods. We generalize this construction to two dimensional case by Tensor product space. In addition, the solution of the system discretized by Galerkin method with modified multiple knot B-spline wavelets is discussed. We also consider a nonlinear partial differential equation for unsteady flows in an open channel called Saint-Venant. Since the solving of this problem by some methods such as finite difference and finite element produce unsuitable approximations specially in the ends of channel, it is solved by multiple knot B-spline wavelet method that yields a very well approximation. Finally, some numerical examples are given to support our theoretical results.

AMS subject classifications: 65T60, 35L60, 35L04

Key words: Galerkin method, semi-orthogonal, B-spline wavelet, multi-resolution analysis, tensor product, hyperbolic partial differential equation, Saint-Venant equations.

1 Introduction

Solving boundary value problems by Galerkin methods leads to very large systems $Ax = b$. Then, for numerical implementation, it is necessary to generate a sparse matrix A. In order to do this, the basis functions with local support are suitable. In particular, orthonormal basis functions with local support decrease the expenses of numerical implementation. However, construction of orthonormal basis functions

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with local support is not easy. Although, Daubechies et al. in [12] have given such basis functions in wavelet space, but there is no still explicit formulas. Then, the scientists have tried to construct semi-orthogonal basis wavelets with local support and explicit formulas. This can be done by multiple knot B-splines (see [9] and [7]). In this work, by multiple knot B-spline functions, we construct the wavelets that satisfy in the (partially) Dirichlet boundary conditions.

Let us first recall the notions of scaling function and multi-resolution analysis as introduced in [16] and [18]. For a function $\phi \in L^2(\Omega)$, let a reference subspace V_0 be generated as the L^2-closure of the linear span of the integer translates of ϕ, namely:

$$V_0 := clos_{L^2}\langle \phi(.-k) : k \in \mathcal{I}_0 \rangle$$

and consider the other subspaces $V_j := clos_{L^2}\langle \phi_{j,k} : k \in \mathcal{I}_j \rangle, j \geq 0$, where $\phi_{j,k} := 2^{j/2}\phi(2^j.-k), j \geq 0, k \in \mathcal{I}_j$, where $\langle F \rangle$ and $\mathcal{I}_j$ denote the space spanned by F and some appropriate set of indices, respectively.

Definition 1.1. *A function $\phi \in L^2(\Omega)$ is said to generate a multi-resolution analysis (MRA) if it generates a nested sequence of closed subspace V_j that satisfy*

(i) $V_0 \subset V_1 \subset \cdots$;

(ii) $clos_{L^2}(\cup_{j\geq 0} V_j) = L^2(\Omega)$;

(iii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iv) $f \in V_j \Longleftrightarrow f(.+2^{-j}) \in V_j \Longleftrightarrow f(2.) \in V_{j+1},\ j \geq 0$;

(v) $\{\phi_{j,k}\}_{k\in\mathcal{I}_j}$ forms a Riesz basis for V_j, i.e.,

there are constants A and B with $0 < A \leq B < \infty$ such that

$$A \sum_{k\in\mathcal{I}_j} |c_k|^2 \leq \| \sum_{k\in\mathcal{I}_j} c_k \phi_{j,k} \|^2_{L^2(\Omega)} \leq B \sum_{k\in\mathcal{I}_j} |c_k|^2$$

independent of j.

If ϕ generates an MRA, then ϕ is called a scaling function. In case different integer translates of ϕ are orthogonal ($\phi(.-k) \perp \phi(.-\tilde{k})$, for $k \neq \tilde{k}$), the scaling function is called an orthogonal scaling function.

Since the subspace V_j are nested, there exists a subspace W_j, such that

$$V_{j+1} = V_j \oplus W_j, \qquad j \in \mathbb{Z},$$

where W_j is some direct summand, not necessarily the orthogonal one. Then, the problem of constructing the spaces W_j means to find a stable system of functions $\Psi_j =$