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A High-Order NVD/TVD-Based Polynomial Upwind Scheme for the Modified Burgers' Equations

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Abstract. A bounded high order upwind scheme is presented for the modified Burgers' equation by using the normalized-variable formulation in the finite volume framework. The characteristic line of the present scheme in the normalized-variable diagram is designed on the Hermite polynomial interpolation. In order to suppress unphysical oscillations, the present scheme respects both the TVD (total variational diminishing) constraint and CBC (convection boundedness criterion) condition. Numerical results demonstrate the present scheme possesses good robustness and high resolution for the modified Burgers' equation.

AMS subject classifications: 65M08, 76M12

Key words: modified Burgers' equation, TVD, NVD, upwind scheme.

1 Introduction

The modified Burgers' equation (MBE) has the form as follows

$$\frac{\partial u}{\partial t}+u^m\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2},\quad a\le x\le b,\quad t\ge t_0, \tag{1.1}$$

where m is a positive integer with $m\ge 1$. The case with $m=1$ is the so-called viscous Burgers' equation which is the fundamental equation in fluid dynamics. The MBE equation possesses the strongly nonlinear terms in the governing equation modeling many practical transport problems such as ion reflection at quasi-perpendicular shocks, nonlinear waves in a medium with low-frequency pumping or absorption, wave processes in thermoelastic media, turbulence transport, transport and dispersion of pollutants in rivers and sediment transport, etc. Recent researches on the

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theoretical analysis of the MBE equation can be found in the references [1–4]. Meanwhile, numerical solutions of the MBE equations were performed by using the collection method [5], the B-spline finite element methods [6,7], the B-spline collocation methods [8–11], the El-Gendi method [12], the Lattice Boltzmann method [13] and the fourth-order compact scheme [14].

One of key issues in the numerical solution of the MBE equation is the discretization of the strongly nonlinear convection term $u^m\partial u/\partial x$. Stable and bounded convection schemes are usually used to guarantee the numerical solutions convergent to the physical solutions. Despite the well-known lower-order schemes, such as the first-order upwind (FOU) and Power-law scheme, are unconditionally bounded and stable, they may often generate unsatisfactory numerical diffusion to smear the computed solutions. To remedy this defect, second-order and higher-order schemes, such as the central difference (CD), second-order upwind (SOU) [15], quadratic upstream interpolation for convective kinematics (QUICK) [16], cubic upwind interpolation (CUI) [17,18] and Lax-Wendroff [19], were proposed for the approximation of the convection terms. However, none of these linear high-order (HO) schemes possess boundedness according to the Godunov's order barrier theorem [20]. They tend to cause unphysical oscillations in the vicinity of steep gradients and discontinuities, which would destroy numerical results and lead to numerical instability.

Combination of a boundedness property with the HO schemes produces the high-resolution (HR) schemes [21, 22]. The HR schemes can provide good resolution of steep gradients and discontinuities without introducing excessive numerical diffusion and unphysical oscillations in the solution. One of the principal boundedness criteria is the total variational diminishing (TVD) constraint proposed by Harten [21]. Based on the TVD constraint, the limiter function presented by Sweby [22] and Roe [20] are introduced to ensure the boundedness of the numerical schemes. Many limiter functions were proposed since then, such as MINMOD by Sweby [22], SUPERBEE by Roe [20] and MUSCL by van Leer [23, 24] and so on. Another significant technique is the convection boundedness criterion (CBC) by Gaskell and Lau [25] by using the normalized variable (NV) formulation of Leonard [26]. Numerical schemes satisfying the CBC is to be of the convection boundedness. From then on, many schemes were presented by using the CBC condition, such SMART [25], CLAM [27], STOIC [28], HOAB [29], WACEB [30], CUBISTA [45] and so on. Further researches by Yu *et al.* [32], Wei *et al.* [29] and Hou *et al.* [33] indicated that the CBC of Gaskell and Lau focused only on the boundedness and paid no attention to any restriction of the accuracy. To remedy this drawback, Wei *et al.* [29] and Hou *et al.* [33] proposed an improved edition named the BAIR condition (Boundedness, Accuracy and Interpolative Reasonableness) to guarantee both boundedness and high accuracy of the convection schemes . The CBC and BAIR are basically used to design the bounded schemes for the incompressible flows and steady problems. In contrast to the TVD schemes, the CBC schemes may be unbounded in simulating some specific problems like the shock tube flows although they work well in the passive scalar problems [34, 35]. In spite of this, the HR schemes can be easily constructed in the normalized variable formulation