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Exact Vibration Solutions of Nonhomogeneous Circular, Annular and Sector Membranes

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Abstract. In this paper, exact vibration frequencies of circular, annular and sector membranes with a radial power law density are presented for the first time. It is found that in general, the sequence of modes may not correspond to increasing azimuthal mode number n. The normalized frequency increases with the absolute value of the power index $|\nu|$. For a circular membrane, the fundamental frequency occurs at $n = 0$ where n is the number of nodal diameters. For an annular membrane, the frequency increases with respect to the inner radius b. When b is close to one, the width $1 - b$ is the dominant factor and the differences in frequencies are small. For a sector membrane, $n - 1$ is the number of internal radial nodes and the fundamental frequency occurs at $n = 1$. Increased opening angle β increases the frequency.

AMS subject classifications: 74.K15, 74.H45

Key words: Membrane, vibration, non-homogeneous, exact, circular, annular, sector

1 Introduction

The solution to the Helmholtz equation governing vibrating membranes is important in the design of drums, speakers, receivers and electromagnetic waveguides. The membrane with a uniform density (or thickness), or joined uniform pieces, has been well researched and documented. In contrast, literature on continuous, nonhomogeneous membrane are rather few, especially those that gave exact solutions. Exact solutions are useful for checking approximate results from numerical or series solutions.

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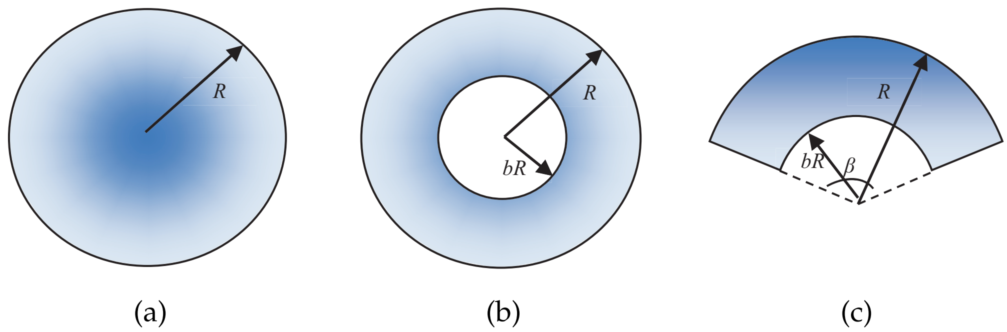

Figure 1: (a) Circular membrane. (b) Annular membrane. (c) Annular sector membrane.

For rectangular membranes, the only solutions seem to be that of Wang [1], who considered a membrane with a linear taper and Wang and Wang [2] who treated membranes with linear density distribution and exponential density distribution. For an annular membrane, De [3] and Wang [1] studied membranes with a density that varies as inverse radius squared, while Gottlieb [4] gave a solution with inverse radius to the fourth power. We mention some papers which considered axisymmetric vibrations only [5, 6]. Such solutions would not give the complete frequency spectrum, since the asymmetric modes are not included.

This paper presents some new exact solutions for the circular or annular membranes and sector membranes whose densities are functions of the radius (see Figs. 1(a) to 1(c)).

2 Problem definition

Consider a membrane with radius R, under uniform stress T_0 and maximum density ρ_0. The normalized membrane equation of motion, in cylindrical coordinates (r,θ) is given by

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\omega^2\rho(r)w=0, \tag{2.1}$$

where w is the transverse deflection, ρ the mass density which is a function of radius and normalized by ρ_0, and ω the circular frequency normalized by $\sqrt{T_0/\rho_0 R^2}$. The boundary conditions are that $w=0$ on the boundaries. The problem at hand is to determine the exact circular frequencies of the aforementioned nonhomogeneous membranes.

3 Power law density distribution

We first consider the normalized density (or thickness) that is described by the following power law

$$\rho = cr^{\nu}, \tag{3.1}$$