Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **10**, No. 4, pp. 1025-1056

DOI: 10.4208/aamm.OA-2017-0203
August 2018

Solving the Navier-Lamé Equation in Cylindrical Coordinates Using the Buchwald Representation: Some Parametric Solutions with Applications

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Received 9 August 2017; Accepted (in revised version) 8 March 2018

Abstract. Using a separable Buchwald representation in cylindrical coordinates, we show how under certain conditions the coupled equations of motion governing the Buchwald potentials can be decoupled and then solved using well-known techniques from the theory of PDEs. Under these conditions, we then construct three parametrized families of particular solutions to the Navier-Lamé equation in cylindrical coordinates. In this paper, we specifically construct solutions having 2π-periodic angular parts. These particular solutions can be directly applied to a fundamental set of linear elastic boundary value problems in cylindrical coordinates and are especially suited to problems involving one or more physical parameters. As an illustrative example, we consider the problem of determining the response of a solid elastic cylinder subjected to a time-harmonic surface pressure that varies sinusoidally along its axis and we demonstrate how the obtained parametric solutions can be used to efficiently construct an exact solution to this problem. We also briefly consider applications to some related forced-relaxation type problems.

AMS subject classifications: 35Q74, 35G35, 74H05, 31B35

Key words: Navier-Lamé equation, cylindrical coordinates, Buchwald representation, Buchwald potentials, exact solutions.

1 Introduction

The Navier-Lamé (NL) equation is the fundamental equation of motion in classical linear elastodynamics [1]. This vector equation, which governs the classical motions of a homogeneous, isotropic and linearly elastic solid, cannot generally be solved using elementary methods (such as separation of variables). Many special methods have however been developed that have proved useful for obtaining particular solutions. One

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of the most well-known and effective methods for generating solutions is the method of potentials, whereby the displacement vector is represented as a combination of one or more scalar and/or vector functions (called potentials). The scalar components of the NL equation constitute a set of coupled PDEs that cannot (except in rare cases) be solved exactly regardless of the coordinate system used. Representations in terms of displacement potentials may, once substituted into the NL equation, yield a system of PDEs that are (in the best case scenario) uncoupled, less coupled, or at least simpler (e.g., decreased order in the derivatives) than the original component equations. The most theoretically well-established representations include the Helmholtz-Lamé, Kovalevshi-Iacovache-Somigliana and (dynamic) Papkovich-Neuber representations [2–4]. Despite having theoretically robust properties, these classical representations can be highly cumbersome when used as analytical tools for solving problems. Other representations have been more recently proposed in studies of anisotropic media [5, 6]. While these representations can be specialized to isotropic media, their analytical efficacy when applied to isotropic problems remains largely unexplored.

One representation that has proven to be analytically effective in anisotropic problems with cylindrical symmetry is the so-called Buchwald representation. This representation was first proposed in 1961 in a paper on the subject of Rayleigh waves in a transversely isotropic medium [7]. (An interesting historical sidebar is that cylindrical coordinates were not actually used in this paper). A number of studies published in the following decades made use of Buchwald's representation expressed in cylindrical coordinates [8, 9], but it was not until the late 1990s that the analytical advantages of doing so were first demonstrated in a series of papers by Ahmad and Rahman [10–12]. The systems of interest in the aforementioned papers (and indeed in almost all papers employing the Buchwald representation) are transversely isotropic. The Buchwald representation is also applicable to problems involving isotropic media [13, 14]. Application of the Buchwald representation to such problems is however rare [15–17].

The Buchwald representation involves three scalar potential functions (see Eq. (2.2) of Section 2). As shown in [15] (and by different means in Section 2 of this paper), the Buchwald representation reduces the original scalar components of the NL equation (which are a set of three coupled PDEs) to a set of two coupled PDEs involving two of the potentials and one separate decoupled PDE involving the remaining potential. By assuming separable product solutions and imposing certain conditions on the axial and temporal parts of the three potentials, we show that the coupled subsystem reduces to a homogeneous linear system, which can be solved by elimination. The elimination procedure generates two independent fourth-order linear PDEs with constant coefficients that can subsequently be solved using fundamental solutions to the two-dimensional Helmholtz equation in polar coordinates. We then construct particular solutions possessing 2π-periodic angular parts and care is taken to consider all allowed parameter values including those yielding degenerate cases. A simultaneous solution to the originally decoupled PDE is also obtained using separation of variables. All three Buchwald potentials can thus be completely determined (under the stipulated conditions) and what follows is the