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A New Two-Grid Method for Expanded Mixed Finite Element Solution of Nonlinear Reaction Diffusion Equations

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Abstract. In the paper, we present an efficient two-grid method for the approximation of two-dimensional nonlinear reaction-diffusion equations using a expanded mixed finite-element method. We transfer the nonlinear reaction diffusion equation into first order nonlinear equations. The solution of the nonlinear system on the fine space is reduced to the solutions of two small (one linear and one non-linear) systems on the coarse space and a linear system on the fine space. Moreover, we obtain the error estimation for the two-grid algorithm. It is showed that coarse space can be extremely coarse and achieve asymptotically optimal approximation as long as the mesh sizes satisfy $h^{k+1}=\mathcal{O}(H^{3k+1})$. An numerical example is also given to illustrate the effectiveness of the algorithm.

AMS subject classifications: 65N30, 65N15, 65M12

Key words: Error estimation, mixed finite elements, reaction-diffusion equations, two-grid methods.

1 Introduction

In this paper, we consider the following nonlinear reaction-diffusion equations

$$sp_t-\nabla\cdot(K(p)\nabla p)=f(x,t),\quad (x,t)\in\Omega\times J, \tag{1.1}$$

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- with initial condition:

$$p(x,0)=p^0(x), \quad x\in\Omega, \tag{1.2}$$

- and boundary condition:

$$(K(p)\nabla p)\cdot\boldsymbol{\nu}=0, \quad (x,t)\in\partial\Omega\times J, \tag{1.3}$$

where $\Omega\subset R^2$ is a polygonal domain with boundary $\partial\Omega$, $\boldsymbol{\nu}$ is the unit exterior normal to $\partial\Omega$, $p_t=\partial p/\partial t$, $J=(0,T]$ and K is a square integrable, symmetric, uniformly positive definite tensor defined on Ω. They often arise from fist-order mass balance and constitutive laws governing a potential p and a related flux $\boldsymbol{u}$:

$$s\frac{\partial p}{\partial t}+\nabla\cdot\boldsymbol{u}=f(x,t), \tag{1.4a}$$

$$K^{-1}(p)\boldsymbol{u}+\nabla p=0. \tag{1.4b}$$

Reaction diffusion equations have received a great deal of attention in numerous applications noteworthy among them being mathematical models of hydrologic and biogeochemical phenomena [15, 19]. In the case, p is the unknown pressure head; s is the compressibility coefficient; $\boldsymbol{u}$ is the Darcy velocity of the water; and $f(p)$ models external flow rate. Mixed finite element methods have been found be very important for solving the ground-water problem. In fact, there is substantial literature on the application in ground-water flow. For example, there are many applications of mixed methods to miscible displacement problems that describe two-phase flow in a petroleum reservoir [14]. In mixed finite element formulation, both the pressure and the flux (Darcy velocity), or displacements and stresses, are approximated simultaneously etc., see [1, 2, 14, 20]. In groundwater hydrology, we have known that the coefficient in the pressure formulation may tend to zero because of low permeability, hence its reciprocal is not readily usable as in standard mixed finite element methods.

To linearize the resulting disctete equations, we use the two-grid method, which was first introduced by Xu [22, 23], as a discretization method for non-symmetric indefinite and non-linear problems. Chen and Huang [7] presented a multilevel iterative method for solving finite element equations of non-linear singular two-point boundary value problems in 1994, Huang and Xue [17] applied the multilevel linearization to the convergence analysis of finite element approximations for Ginzburg-Landau model of D-wave superconductors. Huang [16] constructed a multilevel successive iteration method for non-linear elliptic problems. This approach was applied to non-linear parabolic equations with the expanded mixed finite element method by Dawson and Wheeler [12] and with a finite difference scheme based on the expanded mixed finite element method by Dawson et al [13]. Wu and Allen [21] discussed the case where f is the nonlinear term by the two-grid expanded mixed finite element method. Chen [8–11] made many works of two-grid method for reaction diffusion equations. He and Li [4] considered three iterative methods for solving the stationary Navier-Stokes equations. Hou and his coauthors [5,6]