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Numerical Solution of the Coupled System of Nonlinear Fractional Ordinary Differential Equations

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Abstract. In this paper, we consider the numerical method that is proposed and analyzed in [J. Cao and C. Xu, J. Comput. Phys., 238 (2013), pp. 154–168] for the fractional ordinary differential equations. It is based on the so-called block-by-block approach, which is a common method for the integral equations. We extend the technique to solve the nonlinear system of fractional ordinary differential equations (FODEs) and present a general technique to construct high order schemes for the numerical solution of the nonlinear coupled system of fractional ordinary differential equations (FODEs). By using the present method, we are able to construct a high order schema for nonlinear system of FODEs of the order α, $\alpha>0$. The stability and convergence of the schema is rigorously established. Under the smoothness assumption $f,g\in C^4[0,T]$, we prove that the numerical solution converges to the exact solution with order $3+\alpha$ for $0<\alpha\leq 1$ and order 4 for $\alpha>1$. Some numerical examples are provided to confirm the theoretical claims.

AMS subject classifications: 35R11

Key words: System of fractional ordinary differential equations, high order schema, stability and convergence analysis.

1 Introduction

Fractional differential equations (FDEs) arise in many applied fields as the mathematical modeling of systems and processes (see [20, 29, 33, 36] and references therein). Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than

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the classical models. Fractional ordinary differential equations (FODEs), as generalizations of classical integer order ordinary differential equations, are increasingly used to model problems in fluid flow [33], mechanics, viscoelasticity [21], control theory [32,39], signal processing [27]. Some other related equations have also been widely used in many different fields, such as chaos [45], biology [8,9], biochemistry [44], hydrology [37], finance [18,28,35] and so on. We can see [1,5] for some more recent developments.

The systems of FDEs are different from systems of integer order differential equations, its theoretical investigation is very sparse in the literature. Daftardar-Gejji and Babakhani [12] have presented analysis of system of FDEs, wherein existence and uniqueness theorems for the initial value problem for the system of FDEs have been proved. It is discussed that the existence, uniqueness and stability of solutions of the system of nonlinear fractional differential equations involving Caputo derivatives in [13]. Wang et al. investigated the existence of solutions for systems of nonlinear fractional differential equations by establishing a comparison result and using the monotone iterative technique combined with the method of upper and lower solutions [41]. There has also been a surge in the study of the theory of fractional differential systems and dealing with the solvability of system of nonlinear fractional differential equations [6, 11, 14, 15, 23, 24, 34]. The study of coupled systems involving fractional differential equations is quite important as such systems occur in various problems of applied nature, for instance, see [2–4,7,38,40] and the references therein.

For a general right hand side function, especially the function is nonlinear, it is usually difficult to obtain the analytical solution for systems of fractional differential equations. There are some numerical methods for fractional order PDEs. It is noted that Wang et al. [42,43] develop a fast finite difference method for fractional diffusion equations, these work is very helpful to our research. Thus there is a need to develop numerical methods for system of FODEs. While there exist enormous literatures on the numerical investigation for systems of integer order differential equations, the investigation of numerical methods for system of FODEs are quite limited. In general, there exists no method that yields an exact solution for systems of fractional differential equations. Only approximate solutions can be derived using linearization or perturbation methods. Several methods have been suggested to solve nonlinear system of FDEs. These methods include the homotopy perturbation method [31], Adomian's decomposition method [19] and variation iteration method [30]. However, the convergence region of the corresponding results is rather small. Cao and Xu [10] presented a general technique to construct high order schemes for the numerical solution of the FODEs. This technique is based on the so-called block-by-block approach, which is a common method for the integral equations.

In this paper, we construct and analyze a fractional high order numerical method for system of nonlinear FODEs. The method follows the idea of the block-by-block approach [22], extends the high order schema in [10] to the systems. We are able to derive the stability and error estimates. We prove that convergence order of the schema is $3+\alpha$ for $0<\alpha\leq 1$ and 4 for $\alpha>1$. To our knowledge, this is the first schema of such order with a rigorous convergence proof. Several numerical tests are conducted to support the