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Convergence of Monotone Schemes for Conservation Laws with Zero-Flux Boundary Conditions

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Abstract. We consider a scalar conservation law with zero-flux boundary conditions imposed on the boundary of a rectangular multidimensional domain. We study monotone schemes applied to this problem. For the Godunov version of the scheme, we simply set the boundary flux equal to zero. For other monotone schemes, we additionally apply a simple modification to the numerical flux. We show that the approximate solutions produced by these schemes converge to the unique entropy solution, in the sense of [7], of the conservation law. Our convergence result relies on a BV bound on the approximate numerical solution. In addition, we show that a certain functional that is closely related to the total variation is nonincreasing from one time level to the next. We extend our scheme to handle degenerate convection-diffusion equations and for the one-dimensional case we prove convergence to the unique entropy solution.

AMS subject classifications: 35K65, 35L65, 65M06, 65M08, 65M12

Key words: Degenerate parabolic equation, scalar conservation law, zero-flux boundary condition, monotone scheme, convergence.

1 Introduction

We are interested in an initial-boundary value problem of the form

$$\begin{cases} u_t+\nabla\cdot f(u):=u_t+\sum\limits_{i=1}^{d} f_i(u)_{x_i}=0, & (x_1,\cdots,x_d)\in\Omega, \quad t\in(0,T), \\ f(u(x,t))\cdot\nu=0 & \text{a.e. on } \ \partial\Omega\times(0,T), \\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{1.1}$$

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Here $\Omega=\Pi_{i=1}^d(0,a_i)$ is an open rectangular region in $\mathbb{R}^d$ and ν is the a.e. defined outward unit normal vector to the spatial region Ω. We assume the flux functions $u\mapsto f_i(u)$ are Lipschitz-continuous and satisfy $f_i(0)=f_i(1)=0$, $f_i(u)\ge 0$ for $u\in[0,1]$. We assume that the initial function u_0 satisfies

$$u_0\in L^1(\Omega)\cap \mathrm{BV}(\Omega);\quad u_0(x)\in[0,1],\quad \forall x\in\Omega.$$

The well-posedness of the Cauchy problem corresponding to (1.1) was established by Kružkov [19]. The Dirichlet problem, where the conserved quantity u is specified on the spatial boundary, has also been well understood for a long time [4]. On the other hand, the study of problem (1.1), which specifies zero flux through the spatial boundary, did not begin until more recently.

Problems like (1.1) occur in several applications, including porous media flow, sedimentation processes and road traffic. For example, batch or continuous sedimentation processes are utilized in many industrial applications in which a solid-fluid suspension is separated into its solid and fluid components under the influence of gravity. Relevant models often give rise to hyperbolic (or degenerate parabolic) equations with the zero flux (homogeneous Neumann) boundary condition. For examples of such applications in the one-dimensional setting, see, e.g., [5, 8, 10].

Karlsen, Lie and Risebro [18] proposed a front tracking algorithm for producing approximate solutions to (1.1). For the one-dimensional case, they proved that the front tracking approximations converge to a unique weak solution. Their convergence proof relied on a total variation bound. They also proposed a front tracking algorithm for the multidimensional version of the problem, using dimensional splitting. They did not prove convergence of their multidimensional scheme, the main obstacle being the lack of a total variation bound. The authors of [7] studied the multidimensional version of the problem, allowing for a fairly general boundary (specifically, a regular deformable Lipschitz boundary). They also proposed a definition of L^∞ entropy solution, which we have adopted below for the special case of a rectangular boundary. Bürger, Frid and Karlsen considered a sequence of regularized parabolic problems and proved convergence to a unique entropy solution, using the compactness result of [20]. In both [18] and [7], the authors mentioned the seeming lack of a BV bound in the multidimensional case. A significant extension of the results in [7] to general boundary value problems can be found in Andreianov and Sbihi [3]. These authors consider conservation laws with a general dissipative boundary condition, which includes as particular cases the Dirichlet, Neumann (flux), Robin and obstacle boundary conditions. Well-posedness results for degenerate parabolic problems have been provided by Andreianov and Gazibo [1, 2].

For the Cauchy problem, the theory of monotone schemes has been well established for a long time [11, 15, 21]. For the Cauchy problem again, but with a degenerate diffusion term included, the theory of monotone schemes was addressed more recently [13].

To our knowledge, there are no previous published results on the subject of convergence of finite difference schemes for the zero-flux boundary problem (1.1), even for the important one-dimensional case. However, there is a recent convergence result for an