Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **3**, No. 2, pp. 181-203

DOI: 10.4208/aamm.10-m1018
April 2011

Finite Element θ-Schemes for the Acoustic Wave Equation

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Received 11 March 2010; Accepted (in revised version) 2 June 2010
Available online 5 January 2011

Abstract. In this paper, we investigate the stability and convergence of a family of implicit finite difference schemes in time and Galerkin finite element methods in space for the numerical solution of the acoustic wave equation. The schemes cover the classical explicit second-order leapfrog scheme and the fourth-order accurate scheme in time obtained by the modified equation method. We derive general stability conditions for the family of implicit schemes covering some well-known CFL conditions. Optimal error estimates are obtained. For sufficiently smooth solutions, we demonstrate that the maximal error in the L^2-norm error over a finite time interval converges optimally as $\mathcal{O}(h^{p+1}+\Delta t^s)$, where p denotes the polynomial degree, s=2 or 4, h the mesh size, and Δt the time step.

AMS subject classifications: 65M60, 65M12, 65M15

Key words: Finite element methods, discontinuous Galerkin methods, wave equation, implicit methods, energy method, stability condition, optimal error estimates.

1 Introduction

The efficient and accurate numerical approximation of the wave equations involved in modeling acoustic, elastic or electromagnetic wave propagation is of fundamental importance in many real-life problems. In geophysics, it helps for instance in the interpretation of field data and to predict the damage patterns due to earthquakes. Finite difference methods have been widely used for the simulation of time dependent waves because of their simplicity and their efficiency on structured Cartesian meshes [1, 9, 21, 33]. However, in the presence of heterogeneous media and complex geometry or small geometric futures that require locally refined meshes, their usefulness is somewhat limited.

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Alternatively, finite element methods (FEMs) can easily handle complex geometry and heterogeneous media, and it is easy to incorporate free surface boundary conditions and nonmatching grids. They also have the advantage of local discretization techniques using error indicators. Their extension to high order is straightforward, even in the presence of curved boundaries or material interfaces. Attempts at wave simulation using finite elements have used continuous Galerkin methods [3, 5, 8, 14, 15,24,30,31], discontinuous Galerkin methods [17,18,20,28], and mixed finite element methods [11,13,16,19].

In this paper, we are interested in the finite element approximation of the acoustic wave equation

$$u_{tt} - \nabla \cdot (c^2 \nabla u) = f, \quad \text{in} \;\; \Omega \times J, \tag{1.1}$$

with boundary and initial conditions given by

$$u = 0, \qquad \text{on} \;\; \partial\Omega \times J, \tag{1.2a}$$

$$u|_{t=0} = u^0, \quad \text{in} \;\; \Omega, \tag{1.2b}$$

$$u_t|_{t=0} = v^0, \quad \text{in} \;\; \Omega, \tag{1.2c}$$

where $J{=}(0, T)$ is a finite time interval, $T{>}0$, and Ω is a bounded, convex polygonal domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, with boundary $\partial\Omega$. The (known) source term $f(x, t)$ lies in $L^2(J; L^2(\Omega))$, while $u^0(x){\in}H^1_0(\Omega)$ and $v^0(x){\in}L^2(\Omega)$ are prescribed initial conditions. We assume that the speed of propagation, $c(x)$, is piecewise smooth and satisfies the bounds

$$0 < c_{\min} \leq c(x) \leq c_{\max} < \infty, \quad x \in \bar{\Omega}.$$

The standard weak formulation of problem (1.1)-(1.2c) is stated as follows: find $u{\in}L^2(J; H^1_0(\Omega))$, satisfying (1.2b) and (1.2c), with $u_t{\in}L^2(J; L^2(\Omega))$ and $u_{tt}{\in}L^2(J; H^{-1}(\Omega))$, such that

$$\langle u_{tt}, v\rangle + a(u, v) = (f, v), \quad \forall v \in H^1_0(\Omega), \quad \text{a.e. in} \;\; J. \tag{1.3}$$

Here, the time derivatives are understood in the sense of distributions, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $L^2(\Omega)$, $(\cdot, \cdot)$ is the usual inner product in $L^2(\Omega)$, and $a(\cdot, \cdot)$ is the elliptic bilinear form given by

$$a(u, v) = (c\nabla u, c\nabla v). \tag{1.4}$$

Existence and uniqueness of a solution to the variational problem is proved, for instance, in [23]. It is shown that the weak solution u is continuous in time; that is

$$u \in C^0\big(\bar{J}; H^1_0(\Omega)\big), \qquad u_t \in C^0\big(\bar{J}; L^2(\Omega)\big).$$

This result implies in particular that the initial conditions (1.2b) and (1.2c) are well defined; see Chapter 3 in [23] and Chapter 8 in [27] for more details. Additional regularity assumptions will be made throughout the paper to carry out the convergence