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A-Posteriori Error Estimates for Uniform p-Version Finite Element Methods in Square

Jianwei Zhou[1], Danping Yang[2,*] and Yujie Liu[3]

[1] *Department of Mathematics, Linyi University, Shandong 276005, China*
[2] *Department of Mathematics, East China Normal University, Shanghai 200241, China*
[3] *School of Data and Computational Science, Sun Yat-sen University, Guangzhou, Guangdong 510275, China*

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Abstract. In this work, the *a-posteriori* error indicator with an explicit formula for p-version finite element methods in square is investigated, and its reliable and efficient properties are deduced. Especially, this *a-posteriori* error indicator is determined by the right hand item of the model. We reformulate this *a-posteriori* error indicator with finite coefficients, which can be easily calculated during applications.

AMS subject classifications: 65N30, 65M15

Key words: a-posteriori error indicator, Legendre polynomial, p-version finite element method.

1 Introduction

Due to extensive applications of partial differential equations (PDEs, for short), the corresponding models must be solved with high accurate and efficient numerical methods. The h-version finite element method (h-FEM, for short) has become more and more popular during the last decades. And the p-version finite element method (p-FEM, for short) is a very efficient and high accurate numerical method for solving PDEs. While h-FEM uses refined meshes and fixed polynomials, the p-FEM employs fixed meshes and alternative order of the basis functions. In order to get a numerical solution with acceptable accuracy, p-FEM increases the degree of polynomial bases, if the *a-posteriori* error indicators are bigger than some given criteria (see e.g., [2,3]). There are few work on *a-posteriori* error estimates for p-version and hp-version finite element methods in the literatures, for more details please refer to [10–13, 15–17, 20, 24] and references cited therein. However, *a-posteriori* error indicators with explicit formulae for p-FEM have been less developed

*Corresponding author.
Email: jwzhou@yahoo.com (J. Zhou), dpyang@math.ecnu.edu.cn (D. Yang), liuyujie5@mail.sysu.edu.cn (Y. Liu)

in theoretics. Guo summarized weighted *a-posteriori* error estimations for p-FEM in one dimension in [12]. Chen investigated the convergence of spectral methods and spectral-collocation methods for Volterra integral equations in [7, 22]. Yang discussed *a-posteriori* error estimates for discretized discontinuous Galerkin approximation for reactive transport problems in [23]. The authors presented some improved *a-posteriori* error estimates for Galerkin spectral methods in [24, 25].

This work focuses on studying *a-posteriori* error estimates for uniform p-version finite element methods in square with a fixed mesh. We emphatically declare that the *a-posteriori* error indicator has an explicit formula, which only includes four coefficients of Legendre polynomial expansion of the right-hand item. And hence, with this simple formula, the *a-posteriori* error indicator can be easily used in practical applications.

The remainder of this paper is organized as follows. The model and its p-version finite element methods are presented in Section 2. The *a-posteriori* error indicator and its efficient and reliable properties are deduced in Section 3, specially, the explicit formula of the *a-posteriori* error indicator is obtained. Section 4 contains some numerical examples to confirm the theoretical results. Finally, we state conclusions and the future work in Section 5.

2 The model and its p-version finite element approximation

Let a square $\Omega=(-1,1)\times(-1,1)$ be with the boundary $\partial\Omega=\{|x|=1,\ -1\le y\le 1, \text{ and } |y|=1,\ -1\le x\le 1\}$. Throughout this work, we adopt $W^{m,p}(\Omega)$ for the Sobolev space on Ω as in [1]. In addition, c and C denote generic positive constants. We consider the Poisson equation with a homogeneous Dirichlet boundary condition, which reads

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Let

$$\begin{aligned} (g,z) &= \int_\Omega gz, && \forall g,z\in L^2(\Omega), \\ a(v,w) &= \int_\Omega \nabla v\cdot\nabla w, && \forall v,w\in H^1(\Omega). \end{aligned}$$

As usual, we rewrite the problem (2.1) with a weak formulation: finding $u\in H_0^1(\Omega)$ such that

$$a(u,w)=(f,w), \qquad \forall w\in H_0^1(\Omega). \tag{2.2}$$

Obviously, there exists a unique solution u satisfying (2.2). In this work, with fixed meshes, we solve the model problem with uniform p-version finite element methods, which have the same degree of polynomials on each mesh. For $i=1,2,\cdots,N_\tau+1$, we denote $x_i=2\frac{i-1}{N_\tau}-1$, $I_x^i=(x_i,x_{i+1})$, $h_x^i=|x_{i+1}-x_i|$, $h_x=\max\{h_x^i\}$, and abscissa x can be