Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **9**, No. 1, pp. 186-204

DOI: 10.4208/aamm.2014.m63
February 2017

A Quadratic Triangular Finite Volume Element Method for a Semilinear Elliptic Equation

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Received 21 December 2014; Accepted (in revised version) 5 April 2016

Abstract. In this paper we extend the idea of interpolated coefficients for a semilinear problem to the quadratic triangular finite volume element method. At first we introduce quadratic triangular finite volume element method with interpolated coefficients for a boundary value problem of semilinear elliptic equation. Next we derive convergence estimate in H^1-norm, L^2-norm and L^∞-norm, respectively. Finally an example is given to illustrate the effectiveness of the proposed method.

AMS subject classifications: 65N30

Key words: Semilinear elliptic equation, triangulation, finite volume element with interpolated coefficients.

1 Introduction

The finite volume element method is a discretization technique for solving partial differential equations, especially for those that arise from physical laws including mass, momentum, and energy. The method has been widely used in computational fluid mechanics and other applications because it keeps the mass conservation [2, 5–7, 11, 12, 14, 15, 17, 18, 21, 22, 25–28, 34]. As far as the method is concerned, it is identical to the special case of the generalized difference method or GDM proposed by Li-Chen-Wu [21].

The finite element method with interpolated coefficients is an economic and graceful method. This method was introduced and analyzed for semilinear parabolic problems in Zlamal [35]. Later Larsson-Thomee-Zhang [19] studied the semidiscrete linear triangular finite element with interpolated coefficients and Chen-Larsson-Zhang [10] derived almost optimal order convergence on piecewise uniform triangular meshes by use of superconvergence techniques. Xiong-Chen studied superconvergence of finite element for some semilinear elliptic problems [29–31]. Xiong-Chen first put the interpolation idea

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into the finite volume element method and studied the finite volume element with interpolated coefficients of the two-point boundary problem [32] and the linear triangular finite volume element method for a class of semilinear elliptic equations [33].

Li [20] has considered the finite volume element method for a nonlinear elliptic problem and obtained the error estimate in H^1-norm. Chatzipantelidis-Ginting-Lazarov [8] have studied the finite volume element method for a nonlinear elliptic problem, established the error estimates in H^1-norm, L^2-norm and L^∞-norm. Bi [3] obtains the H^1 and $W^{1,\infty}$ superconvergence estimates between the solution of the finite volume element method and that of the finite element method for a nonlinear elliptic problem. In this paper, we put the excellent interpolating coefficients idea into the finite volume element method on triangular mesh for a semilinear elliptic equation.

We denote Sobolev space and its norm by $W^{k,r}(\Omega)$ and $\|\cdot\|_{k,r}$, respectively [1]. If $r=2$, simply use $H^k(\cdot)$ and $\|\cdot\|_k$ and $\|\cdot\|=\|\cdot\|_0$ is L^2-norm. Further we denote with r' the adjoint of r, i.e.,

$$\frac{1}{r}+\frac{1}{r'}=1, \quad r\geq 1.$$

We assume that the exact solution u is sufficiently smooth for our purpose. Throughout this paper, the constant C denotes different positive constant at each occurrence, which is independent of the mesh size h.

The rest of the paper is organized as follow. First we introduce the quadratic triangular finite volume element method with interpolated coefficients in Section 2 and give preliminaries and some lemmas in Section 3. Next we derive optimal order H^1-norm, L^2-norm and L^∞-norm estimates, respectively, in Section 4. Finally the theoretical results are tested by a numerical example in Section 5.

2 Quadratic finite volume element method with interpolated coefficients

Let $\Omega\subset\mathbb{R}^2$ be a bounded polygonal domain. Consider the second-order semilinear elliptic boundary value problem:

$$\begin{cases} -\Delta u+f(u)=g & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is assumed that $f(s)$ is the sufficiently smooth function with respect to s, and $f'(s)>0$ for finite interval.

Let $V\subset\Omega$ be any control volume with piecewise smooth boundary ∂V. Integrate (2.1) over control volume V, then by the Green's formula, the conservative integral of (2.1) reads, finding u, such that

$$-\int_{\partial V}\frac{\partial u}{\partial n}\mathrm{d}s+\int_V f(u)\mathrm{d}x\mathrm{d}y=\int_V g\mathrm{d}x\mathrm{d}y, \quad V\subset\Omega. \tag{2.2}$$