Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. 7, No. 1, pp. 58-73

DOI: 10.4208/aamm.2013.m128
February 2015

Multi-Symplectic Method for the Zakharov-Kuznetsov Equation

Haochen Li[1,2], Jianqiang Sun[1,*] and Mengzhao Qin[3]

[1] *Department of Mathematics, College of Information Science and Technology, Hainan University, Hainan 570228, China*
[2] *School of Mathematical Science, Nanjing Normal University, Jiangsu 210023, China*
[3] *LSEC, Academy of Mathematics and System Sciences, Chinese Academy of Science, Beijing 100190, China*

Received 24 February 2013; Accepted (in revised version) 13 August 2014

Abstract. A new scheme for the Zakharov-Kuznetsov (ZK) equation with the accuracy order of $\mathcal{O}(\Delta t^2+\Delta x+\Delta y^2)$ is proposed. The multi-symplectic conservation property of the new scheme is proved. The backward error analysis of the new multi-symplectic scheme is also implemented. The solitary wave evolution behaviors of the Zakharov-Kunetsov equation is investigated by the new multi-symplectic scheme. The accuracy of the scheme is analyzed.

AMS subject classifications: 65D17

Key words: The Zakharov-Kuznetsov equation, multi-symplectic method, backward error analysis.

1 Introduction

The two-dimensional generalization of the KDV equation, or the ZK equation

$$u_t+\frac{1}{2}(u^2)_x+u_{xxx}+u_{xyy}=0 \tag{1.1}$$

was first derived by Zakharov and Kuznetsov (1974) [26] in three dimensional form to describe nonlinear ion acoustic waves in a magnetized plasma [13, 16]

$$u_t+uu_x+u_{xxx}+(\Delta u)_x=0, \quad \Delta=\partial_x^2+\partial_y^2+\partial_z^2. \tag{1.2}$$

*Corresponding author.
Email: sunjq123@qq.com (J. Q. Sun)

A variety of physical phenomena, in the purely dispersive limit, are governed by this type of equation; for example, the Rossby waves in rotating atmosphere [22], and the isolated vortex of the drift waves in three dimensional plasma [21]. Although Eq. (1.1) is not even integrable, quite a lot is now known about its nonlinear wave and soliton solutions. Numerical and analytical results of Eq. (1.1) have been investigated in [14,15].

Recently, Chen [9], from the Preissman scheme for multi-symplectic equations, derived a multi-symplectic numerical scheme for the ZK equation that can be simplified to an implicit 36-point scheme. In this paper, we proposed a new multi-symplectic Euler-box scheme to solve the two-dimensional ZK equation.

The paper is organized as follows: in Section 2, the multi-symplectic structure for the ZK equation is introduced and we propose a new multi-symplectic scheme for the ZK equation and prove its discrete multi-symplectic conservation law. In Section 3, we implement the backward error analysis for the new multi-symplectic scheme of the ZK equation. In Section 4, the solitary wave behaviors of the ZK equation are investigated by the new multi-symplectic scheme and the accuracy of the scheme is analyzed. We finish the paper with conclusion remarks in Section 5.

2 A new multi-symplectic scheme for the ZK equation

Introducing the potential $\varphi_x = u$, Eq. (1.1) is equivalent to

$$\varphi_{xxt} + \varphi_x \varphi_{xx} + \varphi_{xxxx} + \varphi_{xxyy} = 0. \tag{2.1}$$

Now, we introduce some variables: $u = \varphi_x$, $v = \varphi_{xx}$, $w = \varphi_{xy}$, $p = -\varphi_{xt}/2$.

According to the covariant De Donder-Weyi Hamilton function theories and the multi-symplectic concept introduced by Bridges [2–7,12], the ZK equation can be reformulated as a system of five first-order partial differential equations which can be written in the form:

$$M\partial_t z + K\partial_x z + L\partial_y z = \nabla_z S(z), \quad z = (p,u,\varphi,v,w)^T \in R^5, \tag{2.2}$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

and $S(z) = up - (v^2 + w^2)/2 - u^3/6$. For details, we refer to [7], $\nabla_z S(z)$ is the gradient of $S(z)$ with respect to the standard inner product on R^5. The system (2.2) is a Hamiltonian formulation of the ZK equation on a multi-symplectic structure, where $M,K,L \in R^{n\times n}$ are skew-symmetric matrices and $S(z): R^n \to R$ is a smooth function of the $z(x,y,t)$.

For Eq. (2.2), one of the most important characteristic is that it satisfies the multi-symplectic conservation law [1,6,7,10,11,18,20,24]

$$\frac{\partial}{\partial t}\omega+\frac{\partial}{\partial x}\kappa+\frac{\partial}{\partial y}q=0, \tag{2.3}$$

where

$$\omega=\frac{1}{2}dz\wedge Mdz, \quad \kappa=\frac{1}{2}dz\wedge Kdz, \quad q=\frac{1}{2}dz\wedge Ldz, \tag{2.4}$$

are differential two-forms. So, when a numerical scheme is developed, we expect that the multi-symplectic conservation law (2.3) should be preserved. Bridges and Reich defined a numerical scheme as a multi-symplectic scheme if the scheme preserves a discrete multi-symplectic conservation law [7]. Specifically, if we discretize Hamiltonian PDEs (2.2) as follows

$$M\partial_t^{i,j,n}z_{i,j}^n+K\partial_x^{i,j,n}z_{i,j}^n+L\partial_y^{i,j,n}z_{i,j}^n=\nabla_z S(z_{i,j}^n), \tag{2.5}$$

where $z_{i,j}^n=z(x_i,y_j,t_n)$, $\partial_t^{i,j,n}$, $\partial_x^{i,j,n}$ and $\partial_y^{i,j,n}$ are the discretizations of the derivatives ∂_t, ∂_x and ∂_y respectively, then the scheme is multi-symplectic provided that it can preserve the following discrete conservation law

$$\partial_t^{i,j,n}\omega_{i,j}^n+\partial_x^{i,j,n}\kappa_{i,j}^n+\partial_y^{i,j,n}q_{i,j}^n=0, \tag{2.6}$$

where

$$\omega_{i,j}^n=\frac{1}{2}(dz_{i,j}^n\wedge Mdz_{i,j}^n), \quad \kappa_{i,j}^n=\frac{1}{2}(dz_{i,j}^n\wedge Kdz_{i,j}^n), \quad q_{i,j}^n=\frac{1}{2}(dz_{i,j}^n\wedge Ldz_{i,j}^n). \tag{2.7}$$

Set t_n, $n=0,1,2,\cdots,N_1$; x_i, $i=1,2,\cdots,N_2$; y_j, $j=1,2,\cdots,N_3$ be the regular grids of the integral domain, $z_{i,j}^n$ is an approximation to $z(x_i,y_j,t_n)$, Δt is the time-step, Δx is the x direction step, Δy is the y direction step, and

$$\delta_{\frac{t}{2}}^{\pm}z_{i,j}^n=\pm\frac{z_{i,j}^{n\pm\frac{1}{2}}-z_{i,j}^n}{\frac{1}{2}\Delta t}, \quad z_{i,j}^{n\pm\frac{1}{2}}=\frac{z_{i,j}^n+z_{i,j}^{n\pm1}}{2}, \quad \delta_x^{\pm}z_{i,j}^n=\pm\frac{z_{i\pm1,j}^n-z_{i,j}^n}{\Delta x}, \quad \delta_x^{\pm}z_{i,j}^n=\pm\frac{z_{i,j\pm1}^n-z_{i,j}^n}{\Delta y}.$$

We propose a new scheme for Eq. (2.2). It can be written as

$$\begin{aligned}M_+\delta_{\frac{t}{2}}^+z_{i,j}^{n+\frac{1}{2}}+M_-\delta_{\frac{t}{2}}^-z_{i,j}^{n+\frac{1}{2}}+K_+\delta_x^+z_{i,j}^{n+\frac{1}{2}}+K_-\delta_x^-z_{i,j}^{n+\frac{1}{2}}&\\ +L_+\delta_y^+z_{i,j}^{n+\frac{1}{2}}+L_-\delta_y^-z_{i,j}^{n+\frac{1}{2}}=\nabla_z S(z_{i,j}^{n+\frac{1}{2}}),&\end{aligned} \tag{2.8}$$

where M_+, M_-, K_+, K_- and L_+, L_- are matrix splitting for the matrices M, K and L, respectively, s.t. [8,17,24],

$$M=M_++M_-, \qquad M_+^T=-M_-, \tag{2.9a}$$

$$K=K_++K_-, \qquad K_+^T=-K_-, \tag{2.9b}$$

$$L=L_++L_-, \qquad L_+^T=-L_-. \tag{2.9c}$$

Scheme (2.8) satisfies the discrete multi-symplectic conservation law.

Theorem 2.1. *The new scheme* (2.8) *is a multi-symplectic scheme with the following discrete multi-symplectic conservation law*

$$\delta_{\frac{t}{2}}^{+}\omega_{i,j}^{n+\frac{1}{2}}+\delta_x^{+}\kappa_{i,j}^{n+\frac{1}{2}}+\delta_y^{+}q_{i,j}^{n+\frac{1}{2}}=0, \tag{2.10}$$

where

$$\omega_{i,j}^{n+\frac{1}{2}}=\frac{1}{2}dz_{i,j}^{n}\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}},\quad \kappa_{i,j}^{n+\frac{1}{2}}=\frac{1}{2}dz_{i-1,j}^{n+\frac{1}{2}}\wedge K_{+}dz_{i,j}^{n+\frac{1}{2}},\quad q_{i,j}^{n+\frac{1}{2}}=\frac{1}{2}dz_{i,j-1}^{n+\frac{1}{2}}\wedge L_{+}dz_{i,j}^{n+\frac{1}{2}}.$$

Proof. Consider the variational equation of (2.8)

$$\begin{aligned}M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}&+M_{-}\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}+K_{+}\delta_x^{+}dz_{i,j}^{n+\frac{1}{2}}+K_{-}\delta_x^{-}dz_{i,j}^{n+\frac{1}{2}}\\&+L_{+}\delta_y^{+}dz_{i,j}^{n+\frac{1}{2}}+L_{-}\delta_y^{-}dz_{i,j}^{n+\frac{1}{2}}=S_{zz}(z_{i,j}^{n+\frac{1}{2}})dz_{i,j}^{n+\frac{1}{2}}.\end{aligned} \tag{2.11}$$

Taking the wedge product with $dz_{i,j}^{n+1/2}$ and the variation equation (2.11), since $dz_{i,j}^{n+1/2}\wedge S_{zz}(z_{i,j}^{n+1/2})dz_{i,j}^{n+1/2}=0$, we have

$$\begin{aligned}dz_{i,j}^{n+\frac{1}{2}}\wedge(M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}&+M_{-}\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}+K_{+}\delta_x^{+}dz_{i,j}^{n+\frac{1}{2}}+K_{-}\delta_x^{-}dz_{i,j}^{n+\frac{1}{2}}\\&+L_{+}\delta_y^{+}dz_{i,j}^{n+\frac{1}{2}}+L_{-}\delta_y^{-}dz_{i,j}^{n+\frac{1}{2}})=0.\end{aligned} \tag{2.12}$$

Considering the items containing $\delta_{t/2}^{+}$ or $\delta_{t/2}^{-}$ in Eq. (2.12), we have

$$\begin{aligned}&dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}+dz_{i,j}^{n+\frac{1}{2}}\wedge M_{-}\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}\\=&dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}+M_{-}^{T}dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}\\=&dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}-M_{+}dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}\\=&dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}\delta_{\frac{t}{2}}^{+}dz_{i,j}^{n+\frac{1}{2}}+\delta_{\frac{t}{2}}^{-}dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}}\\=&dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}\Big[\frac{1}{\frac{1}{2}\Delta t}(dz_{i,j}^{n+1}-dz_{i,j}^{n+\frac{1}{2}})\Big]+\Big[\frac{1}{\frac{1}{2}\Delta t}(dz_{i,j}^{n+\frac{1}{2}}-dz_{i,j}^{n})\Big]\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}}\\=&\frac{1}{\frac{1}{2}\Delta t}(dz_{i,j}^{n+\frac{1}{2}}\wedge M_{+}dz_{i,j}^{n+1}-dz_{i,j}^{n}\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}})\\=&\delta_{\frac{t}{2}}^{+}(dz_{i,j}^{n}\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}}).\end{aligned} \tag{2.13}$$

Considering the items containing δ_x^{+} or δ_x^{-} in Eq. (2.12), we have

$$\begin{aligned}&dz_{i,j}^{n+\frac{1}{2}}\wedge K_{+}\delta_x^{+}dz_{i,j}^{n+\frac{1}{2}}+dz_{i,j}^{n+\frac{1}{2}}\wedge K_{-}\delta_x^{-}dz_{i,j}^{n+\frac{1}{2}}\\=&dz_{i,j}^{n+\frac{1}{2}}\wedge K_{+}\delta_x^{+}dz_{i,j}^{n+\frac{1}{2}}+K_{-}^{T}dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_x^{-}dz_{i,j}^{n+\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&=dz_{i,j}^{n+\frac{1}{2}}\wedge K_+\delta_x^+ dz_{i,j}^{n+\frac{1}{2}}-K_+dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_x^- dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge K_+\delta_x^+ dz_{i,j}^{n+\frac{1}{2}}+\delta_x^- dz_{i,j}^{n+\frac{1}{2}}\wedge K_+dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge K_+\Big[\frac{1}{\Delta x}(dz_{i+1,j}^{n+\frac{1}{2}}-dz_{i,j}^{n+\frac{1}{2}})\Big]+\Big[\frac{1}{\Delta x}(dz_{i,j}^{n+\frac{1}{2}}-dz_{i-1,j}^{n+\frac{1}{2}})\Big]\wedge K_+dz_{i,j}^{n+\frac{1}{2}}\\
&=\frac{1}{\Delta x}(dz_{i,j}^{n+\frac{1}{2}}\wedge K_+dz_{i+1,j}^{n+\frac{1}{2}}-dz_{i-1,j}^{n+\frac{1}{2}}\wedge K_+dz_{i,j}^{n+\frac{1}{2}})\\
&=\delta_x^+(dz_{i-1,j}^{n+\frac{1}{2}}\wedge K_+dz_{i,j}^{n+\frac{1}{2}}).
\end{aligned}\tag{2.14}
$$

Considering the items containing δ_y^+ or δ_y^- in Eq. (2.12), we have

$$
\begin{aligned}
&\ \ dz_{i,j}^{n+\frac{1}{2}}\wedge L_+\delta_y^+ dz_{i,j}^{n+\frac{1}{2}}+dz_{i,j}^{n+\frac{1}{2}}\wedge L_-\delta_y^- dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge L_+\delta_y^+ dz_{i,j}^{n+\frac{1}{2}}+L_-^T dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_y^- dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge L_+\delta_y^+ dz_{i,j}^{n+\frac{1}{2}}-L_+dz_{i,j}^{n+\frac{1}{2}}\wedge\delta_y^- dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge L_+\delta_y^+ dz_{i,j}^{n+\frac{1}{2}}+\delta_y^- dz_{i,j}^{n+\frac{1}{2}}\wedge L_+dz_{i,j}^{n+\frac{1}{2}}\\
&=dz_{i,j}^{n+\frac{1}{2}}\wedge L_+\Big[\frac{1}{\Delta y}(dz_{i,j+1}^{n+\frac{1}{2}}-dz_{i,j}^{n+\frac{1}{2}})\Big]+\Big[\frac{1}{\Delta y}(dz_{i,j}^{n+\frac{1}{2}}-dz_{i,j-1}^{n+\frac{1}{2}})\Big]\wedge L_+dz_{i,j}^{n+\frac{1}{2}}\\
&=\frac{1}{\Delta y}(dz_{i,j}^{n+\frac{1}{2}}\wedge L_+dz_{i,j+1}^{n+\frac{1}{2}}-dz_{i,j-1}^{n+\frac{1}{2}}\wedge L_+dz_{i,j}^{n+\frac{1}{2}})\\
&=\delta_y^+(dz_{i,j-1}^{n+\frac{1}{2}}\wedge L_+dz_{i,j}^{n+\frac{1}{2}}).
\end{aligned}\tag{2.15}
$$

Taking Eqs. (2.13)-(2.15) into Eq. (2.12), we have

$$
\delta_{\frac{t}{2}}^+(dz_{i,j}^n\wedge M_+dz_{i,j}^{n+\frac{1}{2}})+\delta_x^+(dz_{i-1,j}^{n+\frac{1}{2}}\wedge K_+dz_{i,j}^{n+\frac{1}{2}})+\delta_y^+(dz_{i,j-1}^{n+\frac{1}{2}}\wedge L_+dz_{i,j}^{n+\frac{1}{2}})=0. \tag{2.16}
$$

The proof is completed. □

We start from the new scheme (2.8). Note that the matrix splitting (2.9) is not unique [25]. We can obtain different schemes with different splitting methods. Now we take M_+, K_+ and L_+ as upper triangle matrices. They are

$$
M_+=\begin{pmatrix}0&0&0&0&0\\0&0&\frac{1}{2}&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad
K_+=\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad
L_+=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}.
$$

Submitting the above matrices into the multi-symplectic scheme (2.8), we get the discrete form of the multi-symplectic PDEs (2.2)

$$
\delta_x^+\varphi_{i,j}^{n+\frac{1}{2}}=u_{i,j}^{n+\frac{1}{2}}, \tag{2.17a}
$$

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}\varphi_{i,j}^{n+\frac{1}{2}}+\delta_x^{+}v_{i,j}^{n+\frac{1}{2}}+\delta_y^{+}w_{i,j}^{n+\frac{1}{2}}=(p-\frac{1}{2}u^2)_{i,j}^{n+\frac{1}{2}}, \tag{2.17b}$$

$$-\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i,j}^{n+\frac{1}{2}}-\delta_x^{-}p_{i,j}^{n+\frac{1}{2}}=0, \tag{2.17c}$$

$$-\delta_x^{-}u_{i,j}^{n+\frac{1}{2}}=-v_{i,j}^{n+\frac{1}{2}}, \tag{2.17d}$$

$$-\delta_y^{-}u_{i,j}^{n+\frac{1}{2}}=-w_{i,j}^{n+\frac{1}{2}}. \tag{2.17e}$$

Applying δ_x^- to Eq. (2.17b), noting that the finite difference operators mutually commute, we have

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}\delta_x^{-}\varphi_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_x^{+}v_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_y^{+}w_{i,j}^{n+\frac{1}{2}}=\delta_x^{-}p_{i,j}^{n+\frac{1}{2}}-\frac{1}{2}\delta_x^{-}(u_{i,j}^{n+\frac{1}{2}})^2. \tag{2.18}$$

Note that

$$-\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i,j}^{n+\frac{1}{2}}=\delta_x^{-}p_{i,j}^{n+\frac{1}{2}}, \tag{2.19}$$

we have

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}\delta_x^{-}\varphi_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_x^{+}v_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_y^{+}w_{i,j}^{n+\frac{1}{2}}=-\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i,j}^{n+\frac{1}{2}}-\frac{1}{2}\delta_x^{-}(u_{i,j}^{n+\frac{1}{2}})^2. \tag{2.20}$$

Substituting (2.17d) and (2.17e) into (2.20), we have

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}\delta_x^{-}\varphi_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_x^{+}\delta_x^{-}u_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_y^{+}\delta_y^{-}u_{i,j}^{n+\frac{1}{2}}=-\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i,j}^{n+\frac{1}{2}}-\frac{1}{2}\delta_x^{-}(u_{i,j}^{n+\frac{1}{2}})^2. \tag{2.21}$$

If we submit the index i by $i+1$ in Eq. (2.21), then we obtain

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}\delta_x^{-}\varphi_{i+1,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_x^{+}\delta_x^{-}u_{i+1,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_y^{+}\delta_y^{-}u_{i+1,j}^{n+\frac{1}{2}}=-\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i+1,j}^{n+\frac{1}{2}}-\frac{1}{2}\delta_x^{-}(u_{i,j}^{n+\frac{1}{2}})^2. \tag{2.22}$$

Note that

$$\delta_x^{-}\varphi_{i+1,j}^{n+\frac{1}{2}}=\delta_x^{+}\varphi_{i,j}^{n+\frac{1}{2}}=u_{i,j}^{n+\frac{1}{2}}. \tag{2.23}$$

We obtain the following multi-symplectic scheme of the ZK equation:

$$\frac{1}{2}\delta_{\frac{t}{2}}^{+}u_{i,j}^{n+\frac{1}{2}}+\frac{1}{2}\delta_{\frac{t}{2}}^{-}u_{i+1,j}^{n+\frac{1}{2}}+\frac{1}{2}\delta_x^{-}(u_{i+1,j}^{n+\frac{1}{2}})^2+\delta_x^{-}\delta_x^{+}\delta_x^{-}u_{i+1,j}^{n+\frac{1}{2}}+\delta_x^{-}\delta_y^{+}\delta_y^{-}u_{i+1,j}^{n+\frac{1}{2}}=0. \tag{2.24}$$

In finite difference format the scheme is given as follows:

$$\begin{aligned}
&\frac{1}{2\Delta t}(u_{i+1,j}^{n+1}+u_{i,j}^{n+1})+\frac{1}{2(\Delta x)^3}(u_{i+2,j}^{n+1}-3u_{i+1,j}^{n+1}+3u_{i,j}^{n+1}-u_{i-1,j}^{n+1})\\
&\quad+\frac{1}{2\Delta x(\Delta y)^2}(u_{i+1,j+1}^{n+1}-2u_{i+1,j}^{n+1}+u_{i+1,j-1}^{n+1}-u_{i,j+1}^{n+1}+2u_{i,j}^{n+1}-u_{i,j-1}^{n+1})\\
=&\frac{1}{2\Delta t}(u_{i+1,j}^{n}+u_{i,j}^{n})-\frac{1}{2(\Delta x)^3}(u_{i+2,j}^{n}-3u_{i+1,j}^{n}+3u_{i,j}^{n}-u_{i-1,j}^{n})\\
&\quad-\frac{1}{2\Delta x(\Delta y)^2}(u_{i+1,j+1}^{n}-2u_{i+1,j}^{n}+u_{i+1,j-1}^{n}-u_{i,j+1}^{n+1}+2u_{i,j}^{n}-u_{i,j-1}^{n})\\
&\quad-\frac{1}{2\Delta x}\Big(\Big(\frac{u_{i+1,j}^{n+1}+u_{i+1,j}^{n}}{2}\Big)^2-\Big(\frac{u_{i,j}^{n+1}+u_{i,j}^{n}}{2}\Big)^2\Big).
\end{aligned}\tag{2.25}$$

Theorem 2.2. *The discrete multi-symplectic scheme* (2.24) *for the ZK equation* (1.1) *satisfies the discrete multi-symplectic conservation law*

$$\begin{aligned}
\frac{1}{2}\delta_{\frac{t}{2}}^{+}(du_{i,j}^{n}\wedge d\varphi_{i,j}^{n+\frac{1}{2}})+\delta_x^{+}(dp_{i-1,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+\frac{1}{2}}+du_{i-1,j}^{n+\frac{1}{2}}\wedge dv_{i,j}^{n+\frac{1}{2}})&\\
+\delta_y^{+}(du_{i,j-1}^{n+\frac{1}{2}}\wedge dw_{i,j}^{n+\frac{1}{2}})=0.&
\end{aligned}\tag{2.26}$$

Proof. From Eq. (2.8), we can get

$$\begin{aligned}
&\delta_{\frac{t}{2}}^{+}(dz_{i,j}^{n}\wedge M_{+}dz_{i,j}^{n+\frac{1}{2}})+\delta_x^{+}(dz_{i-1,j}^{n+\frac{1}{2}}\wedge K_{+}dz_{i,j}^{n+\frac{1}{2}})+\delta_y^{+}(dz_{i,j-1}^{n+\frac{1}{2}}\wedge L_{+}dz_{i,j}^{n+\frac{1}{2}})\\
=&\frac{1}{2}\delta_{\frac{t}{2}}^{+}(du_{i,j}^{n}\wedge d\varphi_{i,j}^{n+\frac{1}{2}})+\delta_x^{+}(dp_{i-1,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+\frac{1}{2}}+du_{i-1,j}^{n+\frac{1}{2}}\wedge dv_{i,j}^{n+\frac{1}{2}})+\delta_y^{+}(du_{i,j-1}^{n+\frac{1}{2}}\wedge dw_{i,j}^{n+\frac{1}{2}})\\
=&\frac{1}{\Delta t}(du_{i,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+1}-du_{i,j}^{n}\wedge d\varphi_{i,j}^{n+\frac{1}{2}})+\frac{1}{\Delta x}(dp_{i,j}^{n+\frac{1}{2}}\wedge d\varphi_{i+1,j}^{n+\frac{1}{2}}-dp_{i-1,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+\frac{1}{2}}\\
&+du_{i,j}^{n+\frac{1}{2}}\wedge dv_{i+1,j}^{n+\frac{1}{2}}-du_{i-1,j}^{n+\frac{1}{2}}\wedge dv_{i,j}^{n+\frac{1}{2}})+\frac{1}{\Delta y}(du_{i,j}^{n+\frac{1}{2}}\wedge dw_{i,j+1}^{n+\frac{1}{2}}-du_{i,j-1}^{n+\frac{1}{2}}\wedge dw_{i,j}^{n+\frac{1}{2}})\\
=&\frac{1}{2}du_{i,j}^{n+\frac{1}{2}}\wedge\Big(\frac{1}{\frac{1}{2}\Delta t}(d\varphi_{i,j}^{n+1}-d\varphi_{i,j}^{n+\frac{1}{2}})\Big)+\frac{1}{2}\Big(\frac{1}{\frac{1}{2}\Delta t}(du_{i,j}^{n+\frac{1}{2}}-du_{i,j}^{n})\Big)\wedge d\varphi_{i,j}^{n+\frac{1}{2}}\\
&+dp_{i,j}^{n+\frac{1}{2}}\wedge\Big(\frac{1}{\Delta x}(d\varphi_{i+1,j}^{n+\frac{1}{2}}-d\varphi_{i,j}^{n+\frac{1}{2}})\Big)+\Big(\frac{1}{\Delta x}(dp_{i,j}^{n+\frac{1}{2}}-dp_{i-1,j}^{n+\frac{1}{2}})\Big)\wedge d\varphi_{i,j}^{n+\frac{1}{2}}\\
&+du_{i,j}^{n+\frac{1}{2}}\wedge\Big(\frac{1}{\Delta x}(dv_{i+1,j}^{n+\frac{1}{2}}-dv_{i,j}^{n+\frac{1}{2}})\Big)+\Big(\frac{1}{\Delta x}(du_{i,j}^{n+\frac{1}{2}}-du_{i-1,j}^{n+\frac{1}{2}})\Big)\wedge dv_{i,j}^{n+\frac{1}{2}}\\
&+du_{i,j}^{n+\frac{1}{2}}\wedge\Big(\frac{1}{\Delta y}(dw_{i,j+1}^{n+\frac{1}{2}}-dw_{i,j}^{n+\frac{1}{2}})\Big)+\Big(\frac{1}{\Delta y}(du_{i,j}^{n+\frac{1}{2}}-du_{i,j-1}^{n+\frac{1}{2}})\Big)\wedge dw_{i,j}^{n+\frac{1}{2}}\\
=&\frac{1}{2}du_{i,j}^{n+\frac{1}{2}}\wedge\delta_{\frac{t}{2}}^{+}d\varphi_{i,j}^{n+\frac{1}{2}}+\frac{1}{2}\delta_{\frac{t}{2}}^{-}du_{i,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+\frac{1}{2}}+dp_{i,j}^{n+\frac{1}{2}}\wedge\delta_x^{+}d\varphi_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}dp_{i,j}^{n+\frac{1}{2}}\wedge d\varphi_{i,j}^{n+\frac{1}{2}}\\
&+du_{i,j}^{n+\frac{1}{2}}\wedge\delta_x^{+}dv_{i,j}^{n+\frac{1}{2}}+\delta_x^{-}du_{i,j}^{n+\frac{1}{2}}\wedge dv_{i,j}^{n+\frac{1}{2}}+du_{i,j}^{n+\frac{1}{2}}\wedge\delta_y^{+}dw_{i,j}^{n+\frac{1}{2}}+\delta_y^{-}du_{i,j}^{n+\frac{1}{2}}\wedge dw_{i,j}^{n+\frac{1}{2}}.
\end{aligned}\tag{2.27}$$

Differentiating Eqs. (2.17a)-(2.17e) respectively, we can get

$$\delta_x^+ d\varphi_{i,j}^{n+\frac{1}{2}} = du_{i,j}^{n+\frac{1}{2}}, \tag{2.28a}$$

$$\frac{1}{2}\delta_{\frac{t}{2}}^+ d\varphi_{i,j}^{n+\frac{1}{2}} + \delta_x^+ dv_{i,j}^{n+\frac{1}{2}} + \delta_y^+ dw_{i,j}^{n+\frac{1}{2}} = dp_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n+\frac{1}{2}} du_{i,j}^{n+\frac{1}{2}}, \tag{2.28b}$$

$$-\frac{1}{2}\delta_{\frac{t}{2}}^- du_{i,j}^{n+\frac{1}{2}} - \delta_x^- dp_{i,j}^{n+\frac{1}{2}} = 0, \tag{2.28c}$$

$$-\delta_x^- du_{i,j}^{n+\frac{1}{2}} = -dv_{i,j}^{n+\frac{1}{2}}, \tag{2.28d}$$

$$-\delta_y^- du_{i,j}^{n+\frac{1}{2}} = -dw_{i,j}^{n+\frac{1}{2}}. \tag{2.28e}$$

Taking Eqs. (2.28) into Eq. (2.27), we can get

$$\begin{aligned}
&\delta_{\frac{t}{2}}^+ (dz_{i,j}^n \wedge M_+ dz_{i,j}^{n+\frac{1}{2}}) + \delta_x^+ (dz_{i-1,j}^{n+\frac{1}{2}} \wedge K_+ dz_{i,j}^{n+\frac{1}{2}}) + \delta_y^+ (dz_{i,j-1}^{n+\frac{1}{2}} \wedge L_+ dz_{i,j}^{n+\frac{1}{2}}) \\
=& du_{i,j}^{n+\frac{1}{2}} \wedge (dp_{i,j}^{n+\frac{1}{2}} -) + u_{i,j}^{n+\frac{1}{2}} du_{i,j}^{n+\frac{1}{2}} - \delta_x^- dp_{i,j}^{n+\frac{1}{2}} \wedge d\varphi_{i,j}^{n+\frac{1}{2}} + dp_{i,j}^{n+\frac{1}{2}} \wedge du_{i,j}^{n+\frac{1}{2}} \\
&+ \delta_x^- dp_{i,j}^{n+\frac{1}{2}} \wedge d\varphi_{i,j}^{n+\frac{1}{2}} + dv_{i,j}^{n+\frac{1}{2}} \wedge dv_{i,j}^{n+\frac{1}{2}} + dw_{i,j}^{n+\frac{1}{2}} \wedge dw_{i,j}^{n+\frac{1}{2}} \\
=& 0.
\end{aligned} \tag{2.29}$$

The proof is completed. □

3 Backward error analysis for the new multi-symplectic scheme

We now assume z is a sufficiently smooth function that, when evaluated at the lattice points, satisfies Eq. (2.8) [19,23]. Expanding z in a Taylor series about $t_{n+1/2}$, we obtain

$$z_{i,j}^{n+1} = z + \frac{\Delta t}{2} z_t + \frac{1}{2}\Big(\frac{\Delta t}{2}\Big)^2 z_{tt} + \frac{1}{6}\Big(\frac{\Delta t}{2}\Big)^3 z_{ttt} + \frac{1}{24}\Big(\frac{\Delta t}{2}\Big)^4 z_{tttt} + \cdots, \tag{3.1a}$$

$$z_{i,j}^{n} = z - \frac{\Delta t}{2} z_t + \frac{1}{2}\Big(\frac{\Delta t}{2}\Big)^2 z_{tt} - \frac{1}{6}\Big(\frac{\Delta t}{2}\Big)^3 z_{ttt} + \frac{1}{24}\Big(\frac{\Delta t}{2}\Big)^4 z_{tttt} - \cdots, \tag{3.1b}$$

where $z = z(x_i, y_j, t_{n+1/2})$. We have

$$z_{i,j}^{n+\frac{1}{2}} = \frac{z_{i,j}^n + z_{i,j}^{n+1}}{2} = z + \frac{1}{2}\Big(\frac{\Delta t}{2}\Big)^2 z_{tt} + \frac{1}{24}\Big(\frac{\Delta t}{2}\Big)^4 z_{tttt} + \mathcal{O}(\Delta t^5). \tag{3.2}$$

So we have

$$\frac{z_{i,j}^{n+1} - z_{i,j}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} = z_t + \frac{\Delta t^2}{24} z_{ttt} + \mathcal{O}(\Delta t^4), \tag{3.3a}$$

$$\frac{z_{i,j}^{n+\frac{1}{2}} - z_{i,j}^{n}}{\frac{1}{2}\Delta t} = z_t + \frac{\Delta t^2}{24} z_{ttt} + \mathcal{O}(\Delta t^4). \tag{3.3b}$$

From Eq. (3.2), we have

$$z_{i,j}^{n+\frac{1}{2}}=z+\mathcal{O}(\Delta t^2). \tag{3.4}$$

Expanding z in a Taylor series about x_i, we obtain

$$z_{i+1,j}^{n+1}=z_{i,j}^{n+1}+\Delta x(z_x)_{i,j}^{n+1}+\frac{\Delta x^2}{2}(z_{xx})_{i,j}^{n+1}+\cdots, \tag{3.5a}$$

$$z_{i+1,j}^{n}=z_{i,j}^{n}+\Delta x(z_x)_{i,j}^{n}+\frac{\Delta x^2}{2}(z_{xx})_{i,j}^{n}+\cdots. \tag{3.5b}$$

From Eqs. (3.5a) and (3.5b), we have

$$\begin{aligned} z_{i+1,j}^{n+\frac{1}{2}}=\frac{z_{i+1,j}^{n}+z_{i+1,j}^{n+1}}{2}&=z_{i,j}^{n+\frac{1}{2}}+\Delta x(z_x)_{i,j}^{n+\frac{1}{2}}+\frac{\Delta x^2}{2}(z_{xx})_{i,j}^{n+\frac{1}{2}}+\mathcal{O}(\Delta x^3)\\ &=z_{i,j}^{n+\frac{1}{2}}+\Delta x(z_x)+\frac{\Delta x^2}{2}z_{xx}+\mathcal{O}(\Delta x^3)+\mathcal{O}(\Delta x\Delta t^2). \end{aligned}$$

So we can get

$$\frac{z_{i+1,j}^{n+\frac{1}{2}}-z_{i,j}^{n+\frac{1}{2}}}{\Delta x}=z_x+\frac{\Delta x}{2}z_{xx}+\mathcal{O}(\Delta x^2+\Delta t^2). \tag{3.6}$$

In the same way, we can get

$$\frac{z_{i,j}^{n+\frac{1}{2}}-z_{i-1,j}^{n+\frac{1}{2}}}{\Delta x}=z_x-\frac{\Delta x}{2}z_{xx}+\mathcal{O}(\Delta x^2+\Delta t^2), \tag{3.7a}$$

$$\frac{z_{i,j+1}^{n+\frac{1}{2}}-z_{i,j}^{n+\frac{1}{2}}}{\Delta y}=z_y+\frac{\Delta y}{2}z_{yy}+\mathcal{O}(\Delta y^2+\Delta t^2), \tag{3.7b}$$

$$\frac{z_{i,j}^{n+\frac{1}{2}}-z_{i,j-1}^{n+\frac{1}{2}}}{\Delta y}=z_y-\frac{\Delta y}{2}z_{yy}+\mathcal{O}(\Delta y^2+\Delta t^2). \tag{3.7c}$$

Substituting Eqs. (3.3a)-(3.4) and Eqs. (3.6)-(3.7c) into Eq. (2.8) yields the modified PDE

$$M(z_t+\frac{\Delta t^2}{24}z_{ttt})+Kz_x+\frac{\Delta x}{2}(K_+-K_-)z_{xx}+Lz_y+\frac{\Delta y}{2}(L_+-L_-)z_{yy}=\nabla_z S(z). \tag{3.8}$$

Substituting M, K, K_+, K_-, L, L_+, L_- and z into modified Eq. (3.8) gives

$$\varphi_x+\frac{1}{2}\Delta x\varphi_{xx}=u, \tag{3.9a}$$

$$\frac{1}{2}\varphi_t+\frac{1}{48}(\Delta t)^2\varphi_{ttt}+v_x+\frac{1}{2}\Delta x v_{xx}+w_y+\frac{1}{2}\Delta y w_{yy}=p-\frac{1}{2}u^2, \tag{3.9b}$$

$$-\frac{1}{2}u_t-\frac{1}{48}(\Delta t)^2u_{ttt}-p_x+\frac{1}{2}\Delta x p_{xx}=0, \tag{3.9c}$$

$$-u_x+\frac{1}{2}\Delta x u_{xx}=-v, \tag{3.9d}$$

$$-u_y+\frac{1}{2}\Delta y u_{yy}=-w. \tag{3.9e}$$

Substituting Eq. (3.9d) and (3.9e) into Eq. (3.9b), we have

$$p=\frac{1}{2}\varphi_t+\frac{1}{48}(\Delta t)^2\varphi_{ttt}+u_{xx}-\frac{1}{4}(\Delta x)^2u_{xxxx}+u_{yy}-\frac{1}{4}(\Delta y)^2u_{yyyy}+\frac{1}{2}u^2. \tag{3.10}$$

Substituting Eq. (3.10) into Eq. (3.9c), we have

$$\begin{aligned}&-\frac{1}{2}u_t-\frac{1}{48}(\Delta t)^2u_{ttt}-\frac{1}{2}\varphi_{xt}-\frac{1}{48}(\Delta t)^2\varphi_{xttt}-u_{xxx}+\frac{1}{4}(\Delta x)^2u_{xxxxx}-u_{xyy}\\&+\frac{1}{4}(\Delta y)^2u_{xyyyy}-\frac{1}{2}(u^2)_x+\frac{1}{4}\Delta x\varphi_{xxt}+\frac{1}{96}\Delta x(\Delta t)^2\varphi_{xxttt}+\frac{1}{2}\Delta xu_{xxxx}\\&-\frac{1}{8}(\Delta x)^3u_{xxxxxx}+\frac{1}{2}\Delta xu_{xxyy}-\frac{1}{8}\Delta x(\Delta y)^2u_{xxyyyy}+\frac{1}{4}\Delta x(u^2)_{xx}=0.\end{aligned} \tag{3.11}$$

Note that

$$\begin{aligned}&\varphi_{xt}=u_t-\frac{1}{2}\Delta x\varphi_{xxt}, &&\varphi_{xxt}=u_{xt}-\frac{1}{2}\Delta x\varphi_{xxxt},\\&\varphi_{xttt}=u_{ttt}-\frac{1}{2}\Delta x\varphi_{xxttt}, &&\varphi_{xxttt}=u_{xttt}-\frac{1}{2}\Delta x\varphi_{xxxttt},\end{aligned}$$

we have

$$\begin{aligned}&-\frac{1}{2}u_t-\frac{1}{48}(\Delta t)^2u_{ttt}-\frac{1}{2}u_t+\frac{1}{4}\Delta xu_{xt}-\frac{1}{8}(\Delta x)^2\varphi_{xxxt}-\frac{1}{48}(\Delta t)^2u_{ttt}\\&+\frac{1}{96}\Delta x(\Delta t)^2u_{xttt}-\frac{1}{192}(\Delta x)^2(\Delta t)^2\varphi_{xxxttt}-u_{xxx}+\frac{1}{4}(\Delta x)^2u_{xxxxx}-u_{xyy}\\&+\frac{1}{4}(\Delta y)^2u_{xyyyy}-\frac{1}{2}(u^2)_x+\frac{1}{4}\Delta xu_{xt}-\frac{1}{8}(\Delta x)^2\varphi_{xxxt}+\frac{1}{96}\Delta x(\Delta t)^2u_{xttt}\\&-\frac{1}{192}(\Delta x)^2(\Delta t)^2\varphi_{xxxttt}+\frac{1}{2}\Delta xu_{xxxx}-\frac{1}{8}(\Delta x)^3u_{xxxxxx}+\frac{1}{2}\Delta xu_{xxyy}\\&-\frac{1}{8}\Delta x(\Delta y)^2u_{xxyyyy}+\frac{1}{4}\Delta x(u^2)_{xx}=0.\end{aligned} \tag{3.12}$$

So we have

$$\begin{aligned}&u_t+\frac{1}{2}(u^2)_x+u_{xxx}+u_{xyy}\\=&\frac{1}{2}\Delta x\Big(u_t+\frac{1}{2}(u^2)_x+u_{xxx}+u_{xyy}\Big)_x-\frac{1}{24}(\Delta t)^2u_{ttt}-\frac{1}{4}(\Delta x)^2\varphi_{xxxt}\\&+\frac{1}{48}\Delta x(\Delta t)^2u_{xttt}-\frac{1}{96}(\Delta x)^2(\Delta t)^2\varphi_{xxxttt}+\frac{1}{4}(\Delta x)^2u_{xxxxx}\\&+\frac{1}{4}(\Delta y)^2u_{xyyyy}-\frac{1}{8}(\Delta x)^3u_{xxxxxx}-\frac{1}{8}\Delta x(\Delta y)^2u_{xxyyyy}\\=&\frac{1}{2}\Delta x\Big(u_t+\frac{1}{2}(u^2)_x+u_{xxx}+u_{xyy}\Big)_x+\frac{1}{48}\Delta x(\Delta t)^2u_{xttt}\\&-\frac{1}{8}(\Delta x)^3u_{xxxxxx}+\mathcal{O}(\Delta t^2+\Delta x^2+\Delta y^2),\end{aligned} \tag{3.13}$$

which is an $\mathcal{O}(\Delta t^2+\Delta x+\Delta y^2)$ perturbation of the ZK equation (1.1).

The modified equation (3.8) can be written in the form of a standard multi-symplectic PDE

$$\tilde{M}\tilde{z}_t+\tilde{K}\tilde{z}_x+\tilde{L}\tilde{z}_y=\nabla_{\tilde{z}}\tilde{S}(\tilde{z}) \tag{3.14}$$

for $\tilde{z}=(z,z_t,z_{tt},z_x,z_y)^T$, and

$$\tilde{x}=S(z)+\frac{\Delta t^2}{24}z_{tt}^TMz_t-\frac{\Delta x}{2}z_x^TPz_x-\frac{\Delta y}{2}z_y^TQz_y,$$

with the skew-symmetric matrices

$$\tilde{M}=\begin{pmatrix} M & 0 & \frac{\Delta t^2}{24}M & 0 & 0\\ 0 & -\frac{\Delta t^2}{24}M & 0 & 0 & 0\\ \frac{\Delta t^2}{24}M & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\quad \tilde{K}=\begin{pmatrix} K & 0 & 0 & \Delta xP & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -\Delta xP & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{L}=\begin{pmatrix} L & 0 & 0 & 0 & \Delta yQ\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -\Delta yQ & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $P=(K_+-K_-)/2$, $Q=(L_+-L_-)/2$.

4 Numerical simulations

In this section, we test the new derived schemes on the solitary wave of the ZK equation. We consider the ZK equation with exact boundary condition. For fixed n, we give the definition of maxerror(n):

$$\text{maxerror}(n)=\max_{i,j}|u_{i,j}^n-u(x_i,y_j,t_n)|, \tag{4.1}$$

where $u_{i,j}^n$ is the numerical solution while $u(x_i,y_j,t_n)$ is the exact solution.

4.1 Numerical simulation 1

The steady progressive wave solutions of the form $u=U(x-ct,y)$ satisfy the following equation:

$$\Delta U=cU-\frac{1}{2}U^2,\quad \Delta\equiv\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial y^2},\quad X\equiv x-ct, \tag{4.2}$$

where c represents the wave velocity to be determined by solving Eq. (4.2). It is easy to see that a steady progressive wave solution of the form

$$U(x,y,t)=3c\text{sech}^2\Big[\frac{1}{2}\sqrt{c}(X\cos\theta+y\sin\theta)\Big] \tag{4.3}$$

is an exact solution. This solution represents an oblique one-dimensional solitary wave with an inclined angle θ with respect to the x-axis. We carry out our numerical computation on the domain $[0,34]\times[0,2]$ with the parameters $c=2$, $\theta=\pi/3$, and choose $\Delta x=0.2$, $\Delta y=0.1$, $\Delta t=0.1$. We take the following initial conditions

$$U(x,y,5)=3c\text{sech}^2\Big[\frac{1}{2}\sqrt{c}((x-5c)\cos\theta+y\sin\theta)\Big] \tag{4.4}$$

just for computing convenience, and it has nothing to do with the scheme and the results. Fig. 1 shows the initial condition at $t=5$. Figs. 2 and 3 show the numerical solution at $t=8$ and $t=11$ respectively. We can see the moving of wave. Fig. 4 shows the error between the numerical solution and the exact solution at $t=11$. Fig. 5 shows the trend of the maxerror(n) as time evolves. From that, we can see that the scheme has the good numerical performance.

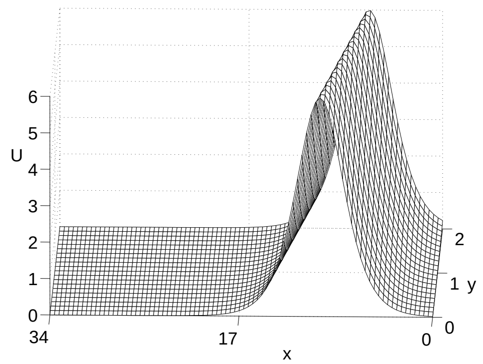

Figure 1: The wave form of the solitary wave at $t=5$.

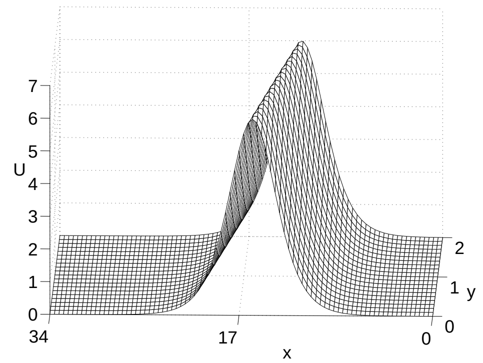

Figure 2: Numerical solution of the solitary wave at $t=8$ with $\Delta x=0.2$, $\Delta y=0.1$ and $\Delta t=0.1$.

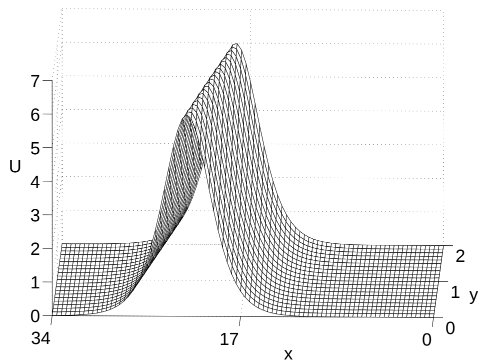

Figure 3: Numerical solution of the solitary wave at $t=11$ with $\Delta x=0.2$, $\Delta y=0.1$ and $\Delta t=0.1$.

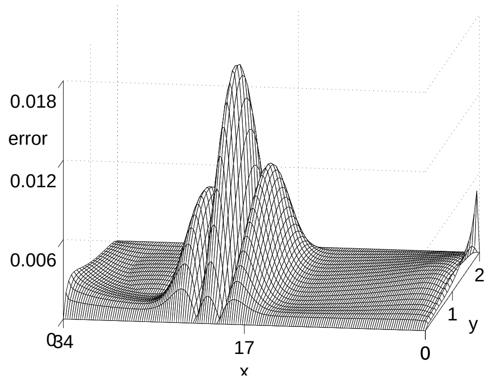

Figure 4: The error between the numerical solution and the exact solution of the solitary wave at $t=11$ with $\Delta x=0.2$, $\Delta y=0.1$ and $\Delta t=0.1$.

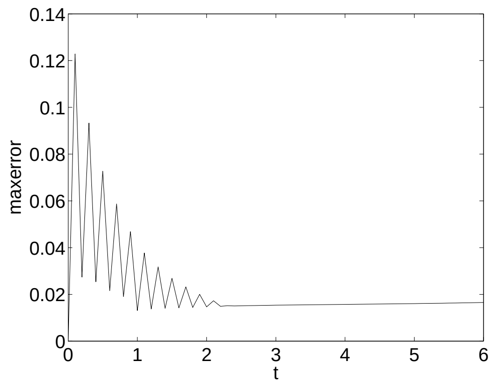

Figure 5: The trend of the maxerror(n) of the solitary wave as time evolves with $\Delta x=0.2$, $\Delta y=0.1$ and $\Delta t=0.1$.

4.2 Numerical simulation 2

Next we try the cylindrical solitary wave of the ZK equation. The cylindrical solition of the ZK equation can be expressed as

$$U(x,y,t)=3\text{sech}^2\Big[\frac{1}{2}\sqrt{(x-ct-x_0)^2+(y-y_0)^2}\Big]. \tag{4.5}$$

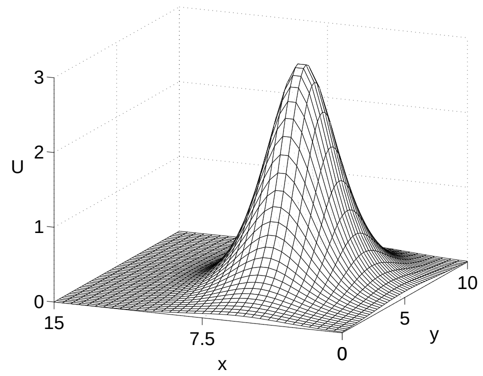

Figure 6: The wave form of the cylindrical solitary wave at $t=0$.

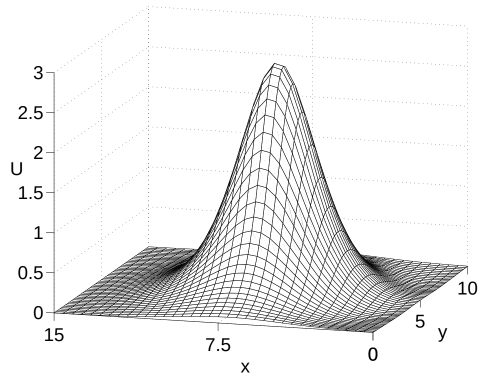

Figure 7: Numerical solution of the cylindrical solitary wave at $t=2.5$ with $\Delta x=0.2$, $\Delta y=0.2$ and $\Delta t=0.1$.

We take the following initial condition:

$$U(x,y,0)=3\text{sech}^2\Big[\frac{1}{2}\sqrt{(x-x_0)^2+(y-y_0)^2}\Big]. \tag{4.6}$$

We compute in a rectangle $[0,15]\times[0,10]$ with the parameters $c=0.5$, $x_0=5.0$, $y_0=5.0$, and choose $\Delta x=0.2$, $\Delta y=0.2$, $\Delta t=0.1$.

Fig. 6 shows the initial condition at $t=0$. Figs. 7 and 8 show the numerical solution at $t=2.5$ and $t=5$ respectively. We can see the moving of wave from the graph. Fig. 9 shows the error between the numerical solution and the exact solution at $t=5$. The error can be diminished by reducing the spacial step and the time step.

5 Conclusions

In this paper, we propose a new scheme for the ZK equation with the accuracy order of $\mathcal{O}(\Delta t^2+\Delta x+\Delta y^2)$. The new scheme is a multi-symplectic scheme that preserves the intrinsic geometry property of the equation. Numerical results show that the new multi-symplectic scheme can well simulate the solitary evolution behaviors of the ZK equation.

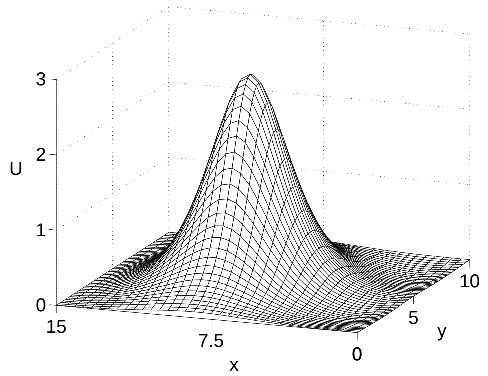

Figure 8: Numerical solution of the cylindrical solitary wave at $t=5$ with $\Delta x=0.2$, $\Delta y=0.2$ and $\Delta t=0.1$.

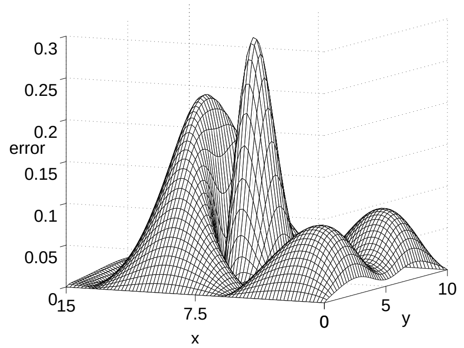

Figure 9: The error between the numerical solution and the exact solution of the cylindrical solitary wave at $t=5$ with $\Delta x=0.2$, $\Delta y=0.2$ and $\Delta t=0.1$.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No. 11161017, 11071251 and 11271195), the Natural Science Foundation of Hainan Province (114003) and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

References

[1] M. J. ABLOWITZ AND C. M. SCHOBER, *Hamiltonian integrators for the nonlinear Schrödinger equation*, Int. J. Modern Phys. C, 5 (1994), pp. 397–401.

[2] T. J. BRIDGES, *Transverse instability of solitary-wave states of the water-wave problem*, J. Fluid Mech., 439 (2001), pp. 255–278.

[3] T. J. BRIDGES, *Universal geometric condition for the transverse instability of solitary waves*, Phys. Rev. Lett., 84 (2000), pp. 2614–2617.

[4] T. J. BRIDGES, *A geometric formulation of the conservation of wave action and its implications for signature and the classification of instabilities*, Proc. Royal. Soc. London A, 453 (1997), pp. 1365–1395.

[5] T. J. BRIDGES AND S. REICH, *Multi-symplectic spectral discretizations for the Zakharov-Kuznetsov and shallow-water equations*, Phys. D, 152 (2001), pp. 491–504.

[6] T. J. BRIDGES AND S. REICH, *Numerical methods for Hamiltonian PDEs*, J. Phys. A Math. Gen., 39 (2006), pp. 5287–5320.

[7] T. J. BRIDGES AND S. REICH, *Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that conserve symplecticity*, Phys. Lett. A, 284 (2001), pp. 184–193.

[8] S. BLANES AND P. C. MOAN, *Splitting methods for the time-dependent Schrödinger equation*, Phys. Lett. A, 265 (2000), pp. 35–42.

[9] J. B. CHEN, *Multisymplectic geometry, local conservation laws and a multisymplectic integrator for the Zakharov-Kuznetsov equation*, Lett. Math. Phys., 63 (2003), pp. 115–124.

[10] J. B. CHEN, *A multisymplectic integrator for the periodic nonlinear Schrodinger equation*, Appl. Math. Comput., 170 (2005), pp. 1394–1417.

[11] K. FENG, Collected Works of Feng Kang II, National Defence Industry Press, Beijing, 1995.

[12] J. FRANK, B. MOORE AND S. REICH, *Linear PDEs and numerical mehods that preserve a multi-symplectic conservation law*, SIAM J. Sci. Comput., 28 (2006), pp. 260–277.

[13] E. INFELD, *Self-focusing of nonlinear ion acoustic waves and solitons in magnetized plasmas*, J. Plasma Phys., 33 (1985), pp. 171–182.

[14] E. INFELD, A. A. SKORUPSKI AND A. SENATORSKI, *Dynamics of waves and multidimensional solitons of the Zakharov-Kuznetsov equation*, J. Plasma Phys., 64 (2000), pp. 397–409.

[15] S. IWASAKI, TOH AND T. KAWAHARA, *Cylindrical qusi-solitons to the Zakharov-Kuznetsov equation*, Phys. D, 43 (1990), pp. 293–303.

[16] E. A. KUZNETSOV, A. M. RUBENCHIK AND V. E. ZAKHAROV, *Soliton stability in plasmas and hydrodynamics*, Phys. Rep., 142 (1986), pp. 103–165.

[17] L. H. KONG, J. L. HONG AND J. J. ZHANG, *Splitting multisymplectic integrators for Maxwell's equations*, J. Comput. Phys., 229 (2010), pp. 4259–4278.

[18] A. AYDIN AND B. KARASOZEN, *Multisymplectic box scheme for the complex modified Korteweg-de Vries equation*, J. Math. Phys., 51 (2010), 083511.

[19] B. MOORE AND S. REICH, *Backward error analysis for multi-symplectic integration methods*, Numerische Mathematik, 95 (2003), pp. 625–652.

[20] J. E. MARSDEN, G. P. PATRICK AND S. SHKOLLER, *Multi-symplectic geometry, variational integrators, and nonlinear PDEs*, Commun. Math. Phys., 199 (1999), pp. 351–395.

[21] K. NOZAKI, *Vortex solitons of drift waves and anomalous diffusion*, Phys. Rev. Lett., 46 (1981), pp. 184–187.

[22] V. I. PETVIASHVILI, *Red spot of Jupiter and the drift soliton in a plasma*, JETP Lett., 32 (1980), pp. 619–622.

[23] S. REICH, *Backward error analysis for numerical integrators*, SIAM J. Numer. Anal., 36 (1999), 1549.

[24] J. Q. SUN AND M. Z. QIN, *Multi-symplectic methods for the coupled 1D nonlinear Schrödinger system*, Comput. Phys. Commun., 155 (2003), pp. 221–235.

[25] Y. S. WANG, B. WANG AND X. CHEN, *Multi-symplectic Euler-box scheme for the KdV equation*, China Phys. Lett., 24 (2007), pp. 312–314.

[26] V. E. ZAKHAROV AND E. A. KUZNETSOV, *Three-dimensional solitons*, Soviet. Phys., 39 (1974), pp. 285–286.