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A Modified Kernel Method for Solving Cauchy Problem of Two-Dimensional Heat Conduction Equation

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Abstract. In this paper, a Cauchy problem of two-dimensional heat conduction equation is investigated. This is a severely ill-posed problem. Based on the solution of Cauchy problem of two-dimensional heat conduction equation, we propose to solve this problem by modifying the kernel, which generates a well-posed problem. Error estimates between the exact solution and the regularized solution are given. We provide a numerical experiment to illustrate the main results.

AMS subject classifications: 47A52, 65J22

Key words: Ill-posed problem, Cauchy problem, modified kernel method.

1 Introduction

In many industrial applications one wants to determine the temperature or heat flux on the surface of a body, where the surface itself is inaccessible for measurements [1]. The Cauchy problem of the heat conduction equation can be considered as a data completion problem that means to achieve the remaining part information from boundary conditions for both the solution and its normal derivative of the boundary. This sort of problem many occur in a large field of practical applications. In a one-dimensional setting this situation can be modelled as the following problem for the heat equation. Determining the temperature $u(x,t)$ for $0<x\leq 1$ from temperature measurements $u(0,t)=g(t)$ and heat flux measurements $u_x(0,t)=0$, when $u(x,t)$ satisfies

$$\begin{cases} u_t(x,t)=u_{xx}(x,t), & t\geq 0, \quad 0<x<1, \\ u(0,t)=g(t), & t\geq 0, \\ u_x(0,t)=0, & t\geq 0, \\ u(x,0)=0, & 0<x<1. \end{cases} \tag{1.1}$$

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Problem (1.1) has been studied by several authors, see for instance [3,10] and also [2].

In this paper, motivated by (1.1), we want to extend problem (1.1) to a Cauchy problem of two-dimensional heat conduction equation in a semi-infinite slab, i.e.,

$$\begin{cases} u_t(x,y,t)=u_{xx}(x,y,t)+u_{yy}(x,y,t), & 0<x<1, \quad y>0, \quad t>0, \\ u(0,y,t)=g(y,t), & y\geq 0, \quad t\geq 0, \\ u_x(0,y,t)=0, & y\geq 0, \quad t\geq 0, \\ u(x,y,0)=0, & 0\leq x\leq 1, \quad y\geq 0, \\ u(x,0,t)=0, & 0\leq x\leq 1, \quad t\geq 0, \\ u(x,y,t)|_{y\to\infty} \text{ bounded}, & 0\leq x\leq 1, \quad t\geq 0. \end{cases} \tag{1.2}$$

Due to the complexity of this problem, it is much more difficult to solve Cauchy problem of heat conduction equation in the 2D case. To the knowledge of the authors, there are still very few results on Cauchy problem of 2D heat conduction problem, e.g., the articles by Li and Wang [9], Qian and Fu [12].

In order to apply the Fourier transform, we extend the functions $u(x,\cdot,\cdot)$ to be whole real (y,t) plane by defining them to be zero everywhere in $\{(y,t),\ y<0,\ t<0\}$. We also assume that these functions are in $L^2(R^2)$. Practically, the input data $g(y,t)$ is measured, there will be measured data function $g^\delta(y,t)\in L^2(R^2)$ with measured error which satisfy

$$\|g^\delta-g\|_{L^2(R^2)}\leq\delta, \tag{1.3}$$

the constant $\delta>0$ represents a bound on the measurement error. Our aim is to seek the solution $u(x,y,t)$ form the Cauchy data $[u,u_x]$, given on the line $x=0$.

It is well-known that Cauchy problem is generally ill-posed, i.e., the existence, uniqueness and stability of their solutions are not always guaranteed, see e.g., Hadamard [8]. A small perturbation in the data $g^\delta(y,t)$ may cause a dramatically large error in the corresponding solution $u(x,y,t)$ for $0<x\leq 1$. Such ill-posedness is caused by the perturbation of high frequencies. Thus, an appropriate regularization method is required.

The paper is organized as follows: in Section 2, we demonstrate ill-posedness of a Cauchy problem of 2D heat conduction equation. In Section 3, we propose a modifying kernel method to solve this ill-posed problem and give error estimates between the regularization solution and the exact solution under a priori choice of the regularization parameter. In Section 4, A numerical experiment is given to illustrate the accuracy and efficiency of our method. Finally, we conclude this paper in Section 5.

2 Ill-posedness of a Cauchy problem of 2D heat conduction equation

Here, and in the following sections, $\|\cdot\|$ denotes the $L^2(R^2)$-norm as

$$\|f\|=\Big(\int_{R^2}|f(y,t)|^2dydt\Big)^{\frac{1}{2}}. \tag{2.1}$$

Let

$$\widehat{f}(\xi,\eta)=\frac{1}{2\pi}\int_{R^2} f(y,t)e^{-i(\xi y+\eta t)}dydt, \quad \xi,\eta\in R,$$

be the Fourier transform of the function $f(y,t)\in L^2(R^2)$. The corresponding inverse Fourier transform of the function $\widehat{f}(\xi,\eta)$ is

$$f(y,t)=\frac{1}{2\pi}\int_{R^2}\widehat{f}(\xi,\eta)e^{i(\xi y+\eta t)}d\xi d\eta.$$

Applying this transformation to problem (1.2) with respect to y and t, we can know

$$\begin{cases} (i\eta)\widehat{u}(x,\xi,\eta)=\widehat{u}_{xx}(x,y,t)-\xi^2\widehat{u}(x,\xi,\eta), & 0<x<1, \quad \xi>0, \quad \eta>0,\\ \widehat{u}(0,\xi,\eta)=g(\xi,\eta), & \xi>0, \quad \eta>0,\\ \widehat{u}(0,\xi,\eta)=0, & \xi>0, \quad \eta>0,\\ \widehat{u}(x,\xi,0)=0, & 0\le x\le 1, \quad \xi\in R,\\ \widehat{u}(x,0,\eta)=0, & 0\le x\le 1, \quad \eta\in R,\\ \widehat{u}(x,\xi,\eta)|_{\xi\to\infty} \text{ bounded}, & 0\le x\le 1, \quad \eta\in R. \end{cases} \tag{2.2}$$

We can obtain the solution of problem (2.2):

$$\widehat{u}(x,\xi,\eta)=\widehat{g}(\xi,\eta)\cosh(\mu x). \tag{2.3}$$

Applying inverse Fourier transformation to (2.3) with respect to ξ and η, we can get the solution of problem (1.2) (see, e.g., [4–6, 12]):

$$u(x,y,t)=\frac{1}{2\pi}\int_{R^2}\widehat{g}(\xi,\eta)\cosh(\mu x)e^{i(\xi y+\eta t)}d\xi d\eta, \tag{2.4}$$

where μ is the principal value of $\sqrt{i\eta+\xi^2}$:

$$\mu=\sqrt{i\eta+\xi^2}=\sqrt{\frac{\sqrt{\eta^2+\xi^4}+\xi^2}{2}}+isign(\eta)\sqrt{\frac{\sqrt{\eta^2+\xi^4}-\xi^2}{2}}.$$

We define $\cosh(\mu x)$ as the "kernel" of an exact solution of problem (1.2). Since $|\cosh(\mu x)|$ is unbounded for $x>0$, small errors in the data can blow up and hardly get any meaningful solution for $0<x\le 1$. Moreover, an error in the high-frequency component is amplified by the factor $\exp\big[\sqrt{(\sqrt{\eta^2+\xi^4}+\xi^2)/2}\big]$, and the kernel includes two variables ξ and η. Comparing this with the Cauchy problem of 1D heat conduction equation, we know that Cauchy problem of 2D heat conduction equation is more ill-posed. Consequently, it is more difficult to compute Cauchy problem of 2D heat conduction equation than Cauchy problem of 1D heat conduction equation.

Following the above analysis, we will construct a regularization method by modifying the kernel $\cosh(\mu x)$.

3 A modifying kernel method and convergence estimates

In this section, we can construct a regularized solution by modifying the kernel, in the present of noisy data, as

$$\widehat{u}_\alpha^\delta(x,\xi,\eta)=\frac{\cosh(\mu x)}{1+\alpha|\cosh(\mu)|^m}\widehat{g}^\delta(\xi,\eta), \tag{3.1}$$

or equivalently,

$$u_\alpha^\delta(x,y,t)=\frac{1}{2\pi}\int_{R^2}\frac{\cosh(\mu x)}{1+\alpha|\cosh(\mu)|^m}\widehat{g}^\delta(\xi,\eta)e^{i(\xi y+\eta t)}d\xi d\eta. \tag{3.2}$$

The basis to do modification is to eliminate all high frequencies or to replace the "kernel" $\cosh(\mu x)$ by a bounded approximation. Note that, if the parameter α is small, $\frac{\cosh(\mu x)}{1+\alpha|\cosh(\mu)|^m}$ is close to $\cosh(\mu x)$, so $\|u_\alpha^\delta-u\|\to 0$. Moreover, for fixed $\alpha>0$, $\frac{\cosh(\mu x)}{1+\alpha|\cosh(\mu)|^m}$ is bounded.

Remark 3.1. This modification can be applied to the general heat conduction problem, see [4–7, 11].

Remark 3.2 (see [12])**.** We could conclude a general form of modified kernel $k(x,\xi,\eta)$, which has the following two common properties:

(I) If the parameter α is small, the kernel $k(x,\xi,\eta)$ is close to $\cosh(\mu x)$.

(II) If α is fixed, $k(x,\xi,\eta)$ is bounded.

Property (I) describes the effect of the kernel $k(x,\xi,\eta)$ closing to $\cosh(\mu x)$ in the low frequency components. Obviously, the smaller the parameter α is, the better the approximation is. Property (II) describes the effect of continuous dependence. Furthermore, the bigger the parameter α is, the better the effect of the regularized solution depends continuously on the data. Consequently, we need a strategy to choose the parameter α in order to keep the balance between the properties (I) and (II). These two properties hint at the regularized role of the parameter α.

The following Lemma is very important to our analysis.

Lemma 3.1 (see [12])**.** *If $a\geq b\geq 0$, $x\geq 0$, $\sigma=sign(\eta)$, $\eta\in R$, we have*

$$|\cosh(a+i\sigma b)|\geq\frac{\sqrt{1-2e^{-\frac{\pi}{2}}}}{2}e^a, \tag{3.3a}$$

$$|\cosh(x(a+i\sigma b))|\leq e^{xa}. \tag{3.3b}$$

The following theorems show that the regularized solution (3.2) is a nice approximation of the exact solution (2.4).

Theorem 3.1. *Let $u(x,y,t)$ be the exact solution of problem* (1.2) *with the exact given data g, and let $u_\alpha^\delta(x,y,t)$ be the new regularized solution* (3.2) *with the noisy data g^δ. Let assumption* (1.3) *be satisfied. Suppose that the exact solution of problem* (1.2) *at $x=1$ has a priori bound*

$$\|u(1,\cdot,\cdot)\|\leq E, \tag{3.4}$$

where the constant $E>0$, if we choose the parameter

$$\alpha=\Big(\frac{\delta}{E}\Big)^m, \tag{3.5}$$

then we have the convergence estimate for fixed $0<x<1$,

$$\|u_\alpha^\delta(x,\cdot,\cdot)-u(x,\cdot,\cdot)\|\leq K\delta^{1-x}E^x. \tag{3.6}$$

Proof. Using Parseval's equality and triangle inequality, we get

$$\begin{aligned}\|u_\alpha(x,\cdot,\cdot)-u_\alpha^\delta(x,\cdot,\cdot)\|&=\|\widehat{u}_\alpha(x,\cdot,\cdot)-\widehat{u}_\alpha^\delta(x,\cdot,\cdot)\|\\&=\Big\|\frac{\cosh(\mu x)}{1+\alpha|\cosh(\mu)|^m}(\widehat{g}^\delta-\widehat{g})\Big\|\\&\leq\delta\sup_{\xi,\eta\in R}B,\end{aligned} \tag{3.7}$$

we set

$$a=\sqrt{\frac{\sqrt{\eta^2+\xi^4}+\xi^2}{2}},\quad b=\sqrt{\frac{\sqrt{\eta^2+\xi^4}-\xi^2}{2}},\quad \sigma=sign(\eta). \tag{3.8}$$

Denote

$$B=\frac{\cosh(\mu x)}{1+\alpha|\cosh(\mu)|^m}. \tag{3.9}$$

From Lemma 3.1, we can obtain

$$B=\frac{\cosh(\mu x)}{1+\alpha|\cosh(\mu)|^m}\leq\frac{e^{xa}}{1+\alpha\big(\frac{1-2e^{-\frac{\pi}{2}}}{4}\big)^{\frac{m}{2}}e^{ma}}. \tag{3.10}$$

Let

$$f(a)=\frac{e^{xa}}{1+\alpha\big(\frac{1-2e^{-\frac{\pi}{2}}}{4}\big)^{\frac{m}{2}}e^{ma}}\quad\text{and}\quad c_1=\Big(\frac{1-2e^{-\frac{\pi}{2}}}{4}\Big)^{\frac{m}{2}},$$

then

$$f'(a)=f(a)\frac{x-\alpha c_1e^{ma}(m-x)}{1+\alpha c_1e^{ma}}. \tag{3.11}$$

It is easy to find

$$a_0=\frac{1}{m}\ln\frac{x}{\alpha c_1(m-x)}, \tag{3.12}$$

such that $f'(a_0)=0$. Moreover, if $a>a_0$, then $f'(a)<0$; if $a<a_0$, then $f'(a)>0$, hence $f(a)$ has a unique maximal value point a_0. Finally,

$$f(a)\le f(a_0)=c_2\alpha^{-\frac{x}{m}}, \tag{3.13}$$

where

$$c_2=\frac{x^{\frac{x}{m}}}{m}c_1^{-\frac{x}{m}}(m-x)^{1-\frac{x}{m}}.$$

So, we know

$$B\le\frac{e^{xa}}{1+\alpha c_1e^{ma}}\le c_2\alpha^{-\frac{x}{m}}. \tag{3.14}$$

Combining (3.7) and (3.14), we have proved the formal stability with respect to perturbation in the data,

$$\|u_\alpha(x,\cdot,\cdot)-u_\alpha^\delta(x,\cdot,\cdot)\|\le c_2\alpha^{-\frac{x}{m}}\delta. \tag{3.15}$$

Moreover, using Parseval's equality, we also get

$$\begin{aligned}\|u_\alpha(x,\cdot,\cdot)-u(x,\cdot,\cdot)\|&=\|\widehat{u}_\alpha(x,\cdot,\cdot)-\widehat{u}(x,\cdot,\cdot)\|\\&=\Big\|\Big(\frac{\cosh(\mu x)}{1+\alpha|\cosh(\mu)|^m}-\cosh(\mu x)\Big)\widehat{g}\Big\|\\&\le\Big\|\frac{\alpha|\cosh(\mu)|^m}{1+\alpha|\cosh(\mu)|^m}\cosh(\mu x)\widehat{g}\Big\|\\&\le E\sup_{\xi,\eta\in R}D.\end{aligned} \tag{3.16}$$

Denote

$$D=\frac{\alpha|\cosh(\mu)|^{m-1}|\cosh(\mu x)|}{1+\alpha|\cosh(\mu)|^m}. \tag{3.17}$$

From Lemma 3.1, we have

$$\begin{aligned}D=&\frac{\alpha|\cosh(\mu)|^{m-1}|\cosh(\mu x)|}{1+\alpha|\cosh(\mu)|^m}\le\frac{\alpha e^{(m-1)a}\cdot e^{xa}}{1+\alpha\big(\frac{1-2e^{-\frac{\pi}{2}}}{4}\big)^{\frac{m}{2}}e^{ma}}\\=&\frac{\alpha e^{(m-1+x)a}}{1+\alpha\big(\frac{1-2e^{-\frac{\pi}{2}}}{4}\big)^{\frac{m}{2}}e^{ma}}.\end{aligned} \tag{3.18}$$

Let

$$p(a)=\frac{e^{(m-1+x)a}}{1+\alpha\big(\frac{1-2e^{-\frac{\pi}{2}}}{4}\big)^{\frac{m}{2}}e^{ma}}\quad\text{and}\quad c_1=\Big(\frac{1-2e^{-\frac{\pi}{2}}}{4}\Big)^{\frac{m}{2}},$$

then

$$p'(a)=p(a)\frac{(1+\alpha c_1e^{ma})(m-1+x)-\alpha c_1e^{ma}m}{1+\alpha c_1e^{ma}}. \tag{3.19}$$

It is easy to find

$$a_1=\frac{1}{m}\ln\Big(\frac{1}{\alpha c_1}\cdot\frac{m-1+x}{1-x}\Big), \tag{3.20}$$

such that $p'(a_1)=0$. Moreover, if $a>a_1$, then $p'(a)<0$; if $a<a_1$, then $p'(a)>0$, hence $p(a)$ has a unique maximal value point a_1. So

$$p(a)\le p(a_1)=c_3\alpha^{-\frac{m-1+x}{m}}, \tag{3.21}$$

where

$$c_3=\frac{(m-1+x)^{\frac{m-1+x}{m}}}{m}c_1^{-\frac{m-1+x}{m}}(1-x)^{1-\frac{m-1+x}{m}}.$$

Then, we can get

$$D\le c_3\alpha^{\frac{1-x}{m}}. \tag{3.22}$$

Combining (3.16) and (3.22), we get the degree of regularized solution approximating exact solution,

$$\|u_\alpha(x,\cdot,\cdot)-u(x,\cdot,\cdot)\|\le c_3\alpha^{\frac{1-x}{m}}E. \tag{3.23}$$

Therefore, combining (3.15) and (3.23), and using the triangle inequality

$$\begin{aligned}\|u^\delta_\alpha(x,\cdot,\cdot)-u(x,\cdot,\cdot)\|\le&\|u^\delta_\alpha(x,\cdot,\cdot)-u_\alpha(x,\cdot,\cdot)\|+\|u_\alpha(x,\cdot,\cdot)-u(x,\cdot,\cdot)\|\\ \le&c_2\alpha^{-\frac{x}{m}}\delta+c_3\alpha^{\frac{1-x}{m}}E\\ =&c_2\delta^{1-x}E^x+c_3\delta^{1-x}E^x\\ =&K\delta^{1-x}E^x,\end{aligned}$$

where $K=c_2+c_3$.

To obtain the continuous dependence of the solution at $x=1$, we need to introduce a stronger a priori assumption instead of (3.4),

$$\|u(1,\cdot,\cdot)\|_p\le E,\quad p>0, \tag{3.24}$$

where $\|\cdot\|_p$ denotes the norm in Sobolev space $H^p(R^2)$ defined by

$$\|u(1,\cdot,\cdot)\|_p=\Big(\int_{R^2}(1+|\xi|^2+|\eta|^2)^p|\widehat{u}(1,\xi,\eta)|^2d\xi d\eta\Big)^{\frac{1}{2}},\quad p>0. \tag{3.25}$$

Thus, we complete the proof. □

Theorem 3.2. *Let $u(1,y,t)$ be the solution of problem* (1.2) *with the exact given data g, and let $u^\delta_\alpha(1,y,t)$ be the new regularized solution of problem* (1.2) *with the noisy data g^δ. Let assumption* (1.3) *be satisfied. Suppose that the solution of problem* (1.2) *at $x=1$ has the a priori bound* (3.24). *Now choosing the parameter*

$$\alpha=\Big(\frac{\delta}{E}\Big)^{\frac{m}{2}}, \tag{3.26}$$

then we have the convergence estimate at $x=1$ for $p>0$,

$$\|u_\alpha^\delta(1,\cdot,\cdot)-u(1,\cdot,\cdot)\|\le c_2\delta^{\frac{1}{2}}E^{\frac{1}{2}}+K_1E\max\Big\{\Big(\frac{1}{6m}\ln\frac{E}{\delta}\Big)^{-p},\Big(\frac{\delta}{E}\Big)^{\frac{3m-1}{6}}\Big\}. \tag{3.27}$$

Proof. Like the proof of Theorem 3.1, we also divide two parts to prove the theorem. For the first part, following the similar process of (3.7)-(3.15). We can easily obtain

$$\|u_\alpha(1,\cdot,\cdot)-u_\alpha^\delta(1,\cdot,\cdot)\|\le c_2\alpha^{-\frac{1}{m}}\delta. \tag{3.28}$$

For the second part, we have

$$\begin{aligned}\|u_\alpha(1,\cdot,\cdot)-u(1,\cdot,\cdot)\|=&\|\widehat{u}_\alpha(1,\cdot,\cdot)-\widehat{u}(1,\cdot,\cdot)\|\\ =&\Big\|\Big(\frac{\cosh(\mu)}{1+\alpha|\cosh(\mu)|^m}-\cosh(\mu)\Big)\widehat{g}\Big\|\\ \le&\Big\|\frac{\alpha|\cosh(\mu)|^m}{1+\alpha|\cosh(\mu)|^m}(1+|\xi|^2+|\eta|^2)^{-\frac{p}{2}}(1+|\xi|^2+|\eta|^2)^{\frac{p}{2}}\cosh(\mu)\widehat{g}\Big\|\\ \le&E\sup_{\xi,\eta\in R}G,\end{aligned} \tag{3.29}$$

where

$$G=\frac{\alpha|\cosh(\mu)|^m}{1+\alpha|\cosh(\mu)|^m}(1+|\xi|^2+|\eta|^2)^{-\frac{p}{2}}. \tag{3.30}$$

From Lemma 3.1, we have

$$G=\frac{\alpha|\cosh(\mu)|^m}{1+\alpha|\cosh(\mu)|^m}(1+|\xi|^2+|\eta|^2)^{-\frac{p}{2}}\le\frac{\alpha e^{ma}}{1+\alpha\big(\frac{1-2e^{-\frac{\pi}{2}}}{4}\big)^{\frac{m}{2}}e^{ma}}(1+|\xi|^2+|\eta|^2)^{-\frac{p}{2}}. \tag{3.31}$$

Let

$$c_1=\Big(\frac{1-2e^{-\frac{\pi}{2}}}{4}\Big)^{\frac{m}{2}}.$$

Now, we distinguish two cases to estimate (3.31).

Case 1: For small a, i.e.,

$$a\le a_2=\frac{1}{m}\ln(\alpha^{-\frac{1}{3m}}),$$

we have

$$G\le\frac{\alpha e^{ma}}{1+\alpha c_1e^{ma}}(1+|\xi|^2+|\eta|^2)^{-\frac{p}{2}}\le\alpha e^{ma_2}\le\alpha^{1-\frac{1}{3m}}. \tag{3.32}$$

Case 2: For large a, i.e., $a>a_2$, we get

$$\begin{aligned}G\le&\frac{\alpha e^{ma}}{1+\alpha c_1e^{ma}}(1+|\xi|^2+|\eta|^2)^{-\frac{p}{2}}\le\frac{1}{c_1}(1+|\xi|^2+|\eta|^2)^{-\frac{p}{2}}\\ \le&\frac{1}{c_1}a_2^{-p}=\frac{1}{c_1}\Big(\frac{1}{m}\ln(\alpha^{-\frac{1}{3m}})\Big)^{-p}.\end{aligned} \tag{3.33}$$

Now using the triangle inequality, and combining (3.28), (3.29), (3.32) with (3.33), we can easily get the convergence estimate

$$\begin{aligned}\|u_\alpha^\delta(1,\cdot,\cdot)-u(1,\cdot,\cdot)\| \le& \|u_\alpha^\delta(1,\cdot,\cdot)-u_\alpha(1,\cdot,\cdot)\|+\|u_\alpha(1,\cdot,\cdot)-u(1,\cdot,\cdot)\| \\ \le& c_2\alpha^{-\frac{1}{m}}\delta+K_1E\max\Big\{\Big(\frac{1}{m}\ln(\alpha^{-\frac{1}{3m}})\Big)^{-p},\alpha^{1-\frac{1}{3m}}\Big\} \\ =& c_2\delta^{\frac{1}{2}}E^{\frac{1}{2}}+K_1E\max\Big\{\Big(\frac{1}{6m}\ln\frac{E}{\delta}\Big)^{-p},\Big(\frac{\delta}{E}\Big)^{\frac{3m-1}{6}}\Big\},\end{aligned}$$

where K_1 is a certain constant. □

Remark 3.3. Here we separately consider the case $0<x<1$ and the case $x=1$. For the case $0<x<1$, the a priori bound for $\|u(1,\cdot,\cdot)\|$ is sufficient. However, for the case $x=1$, the stronger a priori bound for $\|u(1,\cdot,\cdot)\|_p$, where $p>0$ must be imposed. Moreover, through observing previous two theorems, we find that, in the case $0<x<1$ the convergence estimate is Hölder type which fast converge to zero as $\delta\to 0$. However, in the case $x=1$, the convergence estimate just is logarithmic type with the order of $(\ln E/\delta)^{-p}$, $p>0$.

4 Numerical example

Cauchy problem of two-dimensional heat conduction equation is a class of important problems in several engineering contexts and many industrial applications. The physical situation at the surface may be unsuitable for attaching a sensor, or the accuracy of a surface measurement may be seriously impaired by the presence of the sensor. Although it is often difficult to measure accurately the temperature history of the heated surface of a solid, it is easier to measure accurately the temperature history at an interior location of the body.

In previous sections, we have proved that the modified kernel method is stable and convergent with suitable choice of regularization parameter from the theoretical viewpoint. But from the numerical viewpoint, we still need give a concrete example to illustrate the behavior of the proposed method. Here, we assume that u is temperature and consider problem (1.2). Similar to [13], we choose $g(y,t)=e^{-y^2-t^2}$.

In this case the bounded "kernel" is given by $v(x,\xi,\eta)=\cosh(x\sqrt{\xi^2+i\eta})$.

Our numerical procedure for the proposed method is based on the 2D discrete Fourier transform (DFT) and computed with a fast Fourier transform (FFT) algorithm. In using the DFT, it is assumed that the sequence to be transformed is periodic. Thus, we shall make the data vector periodic before computation.

In our numerical implementations, we give the data $g(y,t)$ and sample at an equidistant grid in the domain $[-10,10]\times[-10,10]$ with 64×64 grid points, then carry out a 2D DFT. The discrete noisy data $g^\delta(y,t)$ is obtained by adding a random noise to the exact data $g(y,t)$,

$$g^\delta(y,t)=g(y,t)+\varepsilon\ randn(size(g(y,t))), \tag{4.1}$$

where,

$$\begin{aligned} &g(y,t)=(g(y_1,t_1),g(y_2,t_2),\cdots,g(y_n,t_n))^T,\\ &y_i=-10+(i-1)\Delta y,\quad \Delta y=\frac{20}{n-1},\quad i=1,2,\cdots,n,\\ &t_j=-10+(j-1)\Delta t,\quad \Delta t=\frac{20}{n-1},\quad j=1,2,\cdots,n. \end{aligned}$$

Then the total noise δ can be measured in the sense of Root Mean Square Error according to

$$\delta:=\|g^\delta(y,t)-g(y,t)\|_{l^2}=\sqrt{\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n(g_i^\delta(y_i,t_j)-g_i(y_i,t_j))^2}. \tag{4.2}$$

The function $''randn(\cdot)''$ generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2=1$ and standard deviation $\sigma=1$, $''randn(size(g(y,t)))''$ returns an array of array of random entries that is the same size as $g(y,t)$.

Example 4.1. We choose

$$g(y,t)=e^{-y^2-t^2}. \tag{4.3}$$

Our tests about this example correspond to Figs. 1-2. The numerical results for u and u_α^δ for $m=1$, $m=2$ and $m=3$ are shown in Fig. 1. Fig. 1 shows that the regularization solution approximates the exact solution better as the amount of m increases. This is consistent with Remark 3.2.

In Fig. 2, we make error comparison by plotting comparative curves along x. The numerical results for u and u_α^δ for $\varepsilon=0.01$, $\varepsilon=0.001$ and $\varepsilon=0.0001$ are shown in Fig. 2. Fig. 2 shows that the smaller the ε is, the better the computed solution is. Moreover, we can see that numerical results become worse when x approaches to 1. Physically, the more close to the surface of a solid, the worse the effect.

Here, we point out that Figs. 1-2 verify the stability of our proposed method.

5 Conclusions

In this paper, in order to deal with a Cauchy problem of 2D heat conduction equation, we propose a new regularization method by modifying the kernel and prove the convergence estimates in the whole domain, i.e., including the case $0<x<1$ and the case $x=1$. A numerical example shows that the proposed method works well. Here, a Cauchy problem of 2D heat conduction equation with only nonhomogeneous Dirichlet data on the boundary is solved. However, nonhomogeneous Neumann data is never proposed in this article. We will consider a Cauchy problem of 2D heat conduction equation with both nonhomogeneous Dirichlet data and nonhomogeneous Neumann data on the boundary in the future.

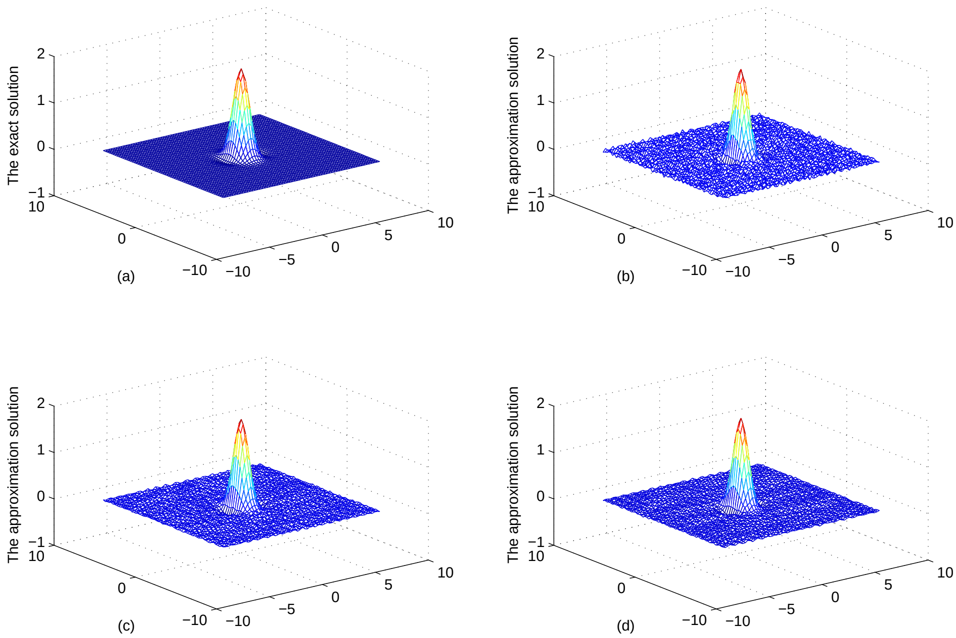

Figure 1: The exact solution u and the approximation solution u_α^δ, $x=0.8$: (a) The exact solution; (b) The approximation solution for $m=1$; (c) The approximation solution for $m=2$; (d) The approximation solution for $m=3$.

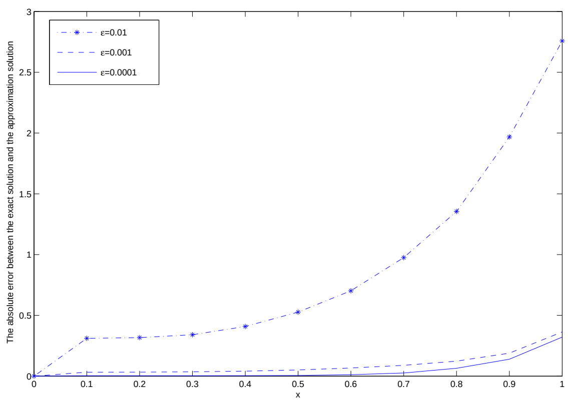

Figure 2: The absolute error between the exact solution u and the approximation solution u_α^δ for the interval $[0,1]$: $\varepsilon=0.01,0.001,0.0001$.

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