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Multiquadric Finite Difference (MQ-FD) Method and its Application

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Abstract. The conventional finite difference (FD) schemes are based on the low order polynomial approximation in a local region. This paper shows that when the polynomial approximation is replaced by the multiquadric (MQ) function approximation in the same region, a new FD method, which is termed as MQ-FD method in this work, can be developed. The paper gives analytical formulas of the MQ-FD method and carries out a performance study for its derivative approximation and solution of Poisson equation and the incompressible Navier-Stokes equations. In addition, the effect of the shape parameter c in MQ on the formulas of the MQ-FD method is analyzed. Derivative approximation in one-dimensional space and Poisson equation in two-dimensional space are taken as model problems to study the accuracy of the MQ-FD method. Furthermore, a lid-driven flow problem in a square cavity is simulated by the MQ-FD method. The obtained results indicate that this method may solve the engineering problem very accurately with a proper choice of the shape parameter c.

AMS subject classifications: 41A10, 41A30, 65N05

Key words: MQ–FD method, shape parameter, central FD method.

1 Introduction

Finite difference (FD) schemes are the most popular approaches used in engineering. In its most general form, the FD method is based on approximating some derivative of a function u at a given point by using a linear combination of the values of u at some surrounding points. Basically, the generation of the finite difference schemes is based on the polynomial approximation. Apart from polynomials, there are a lot of

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other approximate functions such as radial basis functions (RBFs) that can be used to generate finite difference schemes. RBFs are a primary tool for interpolating multidimensional scattered data. Due to their "mesh-free" nature, in the past decade, RBFs have received an increasing attention for derivative approximation and solution of partial differential equations (see, e.g., [1–8]). However, most of these methods are actually based on the function approximation by a global collocation approach. The global collocation approach generally results in a large, ill-conditioned linear system. Furthermore, function approximation approach is very complicated for solving nonlinear problems. These may be the reasons why the method has not so far been extensively applied to solve practical problems.

To resolve these problems and make RBF methods more feasible in solving PDEs, a local method named "local radial basis function-based differential quadrature method" has recently been proposed by Shu et al. [9]. This method adopted the idea of direct approximation of derivative through the differential quadrature (DQ) method, thus can be consistently well applied to linear and nonlinear problems. The DQ method was first proposed by Bellman et al. [10, 11] and its essence is that the derivatives of unknown function can be approximated in terms of the function values at a set of points, either uniformly or non-uniformly distributed. Suppose that a function $f(x)$ is sufficiently smooth, then its mth order derivative with respect to x at a point x_i can be approximated by DQ as

$$\frac{\partial^m f}{\partial x^m}\bigg|_{x_i} = \sum_{j=1}^{N} w_{i,j}^{(m)} f(x_j), \tag{1.1}$$

where x_j are the discrete points in the domain, $f(x_j)$ and $w_{i,j}^{(m)}$ are the function values at these points and the related weighting coefficients. This definition is actually similar to that of the finite difference method, so we can consider the DQ method as a "special" finite difference method. The key to the DQ method is to determine the weighting coefficients in derivative discretization of various orders. In the local RBF-DQ method, based on the analysis of a linear vector space and function approximation, RBFs are taken as the test functions in the DQ approximation to compute the weighting coefficients. Therefore, this method bears both the advantages of RBF approximation, e.g., mesh-free nature, and the advantages of DQ discretization, such as, easy implementation for both linear and nonlinear problems.

In implementing the local RBF-DQ method to solve fluid flow problems, we only need to substitute a set of RBF base functions into Eq. (1.1) and numerically solve the resultant linear equations to obtain the weighting coefficients. Although the procedure is quite simple, we cannot get its analytical formulas for derivative approximation. As a result, it is difficult to theoretically analyze this scheme, such as the influence of shape parameter. In addition, it is very difficult to compare this meshless method with the conventional numerical methods, such as finite difference scheme. In this paper, we apply the idea of local RBF-DQ method to the stencil of the central difference scheme to derive the new MQ-FD method, which has analytical form so that it can be compared with the conventional central difference scheme. In the paper,