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Numerical Approximation of Oscillatory Solutions of Hyperbolic-Elliptic Systems of Conservation Laws by Multiresolution Schemes

Stefan Berres[1], Raimund Bürger[2,*] and Alice Kozakevicius[3]

[1] *Departamento de Ciencias Matemáticas y Físicas, Universidad Católica de Temuco, Temuco, Chile*

[2] *CI²MA and Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile*

[3] *Departamento de Matemática-CCNE, Universidade Federal de Santa Maria, Av. Roraima, 1000, Campus Universitário, Santa Maria, RS, CEP 97105-900, Brazil*

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Abstract. The generic structure of solutions of initial value problems of hyperbolic-elliptic systems, also called mixed systems, of conservation laws is not yet fully understood. One reason for the absence of a core well-posedness theory for these equations is the sensitivity of their solutions to the structure of a parabolic regularization when attempting to single out an admissible solution by the vanishing viscosity approach. There is, however, theoretical and numerical evidence for the appearance of solutions that exhibit persistent oscillations, so-called oscillatory waves, which are (in general, measure-valued) solutions that emerge from Riemann data or slightly perturbed constant data chosen from the interior of the elliptic region. To capture these solutions, usually a fine computational grid is required. In this work, a version of the multiresolution method applied to a WENO scheme for systems of conservation laws is proposed as a simulation tool for the efficient computation of solutions of oscillatory wave type. The hyperbolic-elliptic 2×2 systems of conservation laws considered are a prototype system for three-phase flow in porous media and a system modeling the separation of a heavy-buoyant bidisperse suspension. In the latter case, varying one scalar parameter produces elliptic regions of different shapes and numbers of points of tangency with the borders of the phase space, giving rise to different kinds of oscillation waves.

AMS subject classifications: 76T20, 35L65, 65M06, 76M20, 35M10, 35R25

Key words: Hyperbolic-elliptic system, conservation law, oscillation wave, numerical simulation, multiresolution method, sedimentation model.

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*Corresponding author.
URL: `http://www.ing-mat.udec.cl/~rburger/`
Email: sberres@uct.cl (S. Berres), rburger@ing-mat.udec.cl (R. Bürger), alicek@smail.ufsm.br (A. Kozakevicius)

1 Introduction

1.1 Scope of the paper

We consider first-order systems of two scalar, nonlinear, strongly coupled conservation laws

$$\partial_t \phi_i + \partial_x f_i(\phi_1, \phi_2) = 0, \quad i = 1, 2, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

subject to the initial condition

$$\phi_i(x, 0) = \phi_{i,0}(x), \quad i = 1, 2, \quad x \in \mathbb{R}. \tag{1.2}$$

We recall that the system (1.1) is called *hyperbolic* at a point (ϕ_1, ϕ_2) if the Jacobian $\mathcal{J}_f$ of the flux vector

$$\boldsymbol{f} = (f_1, f_2)^{\mathrm{T}},$$

evaluated at (ϕ_1, ϕ_2),

$$\mathcal{J}_f(\phi_1, \phi_2) := \big(J_{ij}(\phi_1, \phi_2)\big)_{i,j=1,2} := \left(\frac{\partial f_i}{\partial \phi_j}(\phi_1, \phi_2)\right)_{i,j=1,2},$$

has real eigenvalues, that is, if the discriminant

$$\Delta(\phi_1, \phi_2) := \Big((J_{11} - J_{22})^2 + 4J_{12}J_{21}\Big)(\phi_1, \phi_2), \tag{1.3}$$

is non-negative, and *strictly hyperbolic* if these eigenvalues are moreover distinct, that is, if $\Delta(\phi_1, \phi_2) > 0$. If $\Delta(\phi_1, \phi_2) < 0$, then $\mathcal{J}_f(\phi_1, \phi_2)$ has a pair of complex conjugate eigenvalues and (1.1) is called *elliptic* at that point. The set of all points (ϕ_1, ϕ_2) at which (1.1) is elliptic is called *elliptic region*.

It is the purpose of this contribution to study numerically the solution behaviour of (1.1), (1.2) when the initial data are chosen as a pair of constants from the elliptic region, and where these constants are slightly perturbed on a small interval (corresponding to a small number of cells of a spatial discretization). For this setting, and a particular system, Frid and Liu [22] observed highly oscillatory but strongly localized solutions, which they termed *oscillation waves*. We herein capture, and in part analyze, such oscillations first for the system studied in [22] (Model 1), and then for a hyperbolic-elliptic system that emerges from a model of sedimentation of a bidisperse suspension (Model 2) [6, 11]. The novelty of our approach is that we employ a multiresolution (MR) method, which adaptively concentrates computational effort associated with a given numerical scheme for systems of conservation laws on areas of strong variation of the solution. In our case, the method can be advantageously employed to capture the oscillations due to the mixed-type nature of the system.