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A Note on Exact Solutions to Linear Differential Equations by the Matrix Exponential

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Abstract. It is known that the solution to a Cauchy problem of linear differential equations:

$$x'(t) = A(t)x(t), \quad \text{with} \quad x(t_0) = x_0,$$

can be presented by the matrix exponential as $\exp(\int_{t_0}^t A(s)\,ds)x_0$, if the commutativity condition for the coefficient matrix $A(t)$ holds:

$$\Big[\int_{t_0}^t A(s)\,ds, A(t) \Big] = 0.$$

A natural question is whether this is true without the commutativity condition. To give a definite answer to this question, we present two classes of illustrative examples of coefficient matrices, which satisfy the chain rule

$$\frac{d}{dt} \exp\Big(\int_{t_0}^t A(s)\,ds\Big) = A(t) \exp\Big(\int_{t_0}^t A(s)\,ds\Big),$$

but do not possess the commutativity condition. The presented matrices consist of finite-times continuously differentiable entries or smooth entries.

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1 Introduction

It is known that one of the important problems in the theory of differential equations is how to solve Cauchy problems of linear differential equations [1]. If the fundamental matrix solution is found, unique solutions to the Cauchy problems of linear differential equations can be automatically presented.

However, on one hand, it is not always possible to compute the fundamental matrix solution explicitly. On the other hand, linear differential equations are also used in solving nonlinear integrable equations, in both continuous and discrete cases [2,3]. Therefore, the explicit representation of solutions to the Cauchy problems of linear differential equations is a crucial issue in the theory of both linear and nonlinear differential equations.

Let us specify a system of linear differential equations on an interval $I=(a,b)\subseteq\mathbb{R}$ as follows:

$$x'(t) = A(t)x(t) + f(t), \tag{1.1}$$

where $f(t)\in\mathbb{R}^n$ is continuous on I and $A(t)$ is an $n \times n$ matrix of real continuous functions on I. Any higher-order scalar linear differential equation of Kovalevskaia type can be transformed into the above linear system. Of significant importance in the theory of differential equations is how to solve the Cauchy problem on I:

$$x'(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0,$$

where $t_0\in I$ and $x_0\in\mathbb{R}^n$ are given. Various examples of finding solutions to Cauchy problems of differential equations, both linear and nonlinear, can be found in [1,4].

If the coefficient matrix $A(t)$ commutes with its integral $\int_{t_0}^t A(s)\,ds$:

$$[A(t), \int_{t_0}^t A(s)\,ds] = 0, \quad t \in I, \tag{1.2}$$

then the fundamental matrix solution $U(t,t_0)$ of the homogeneous system $x'(t)= A(t)x(t)$ is determined by the matrix exponential (see, say, [5]):

$$U(t,t_0) = \exp \int_{t_0}^t A(s)\,ds, \quad t \in I. \tag{1.3}$$

That is to say, if we have the commutativity condition (1.2), the following chain rule holds:

$$\frac{d}{dt} \exp \int_{t_0}^t A(s)\,ds = A(t) \exp \int_{t_0}^t A(s)\,ds, \quad t \in I, \tag{1.4}$$

and so, the unique solution to the Cauchy problem (1) is given by the variation of parameters formula:

$$x(t) = U(t,t_0)x_0 + \int_{t_0}^t U(t,s)f(s)\,ds, \quad t \in I. \tag{1.5}$$