Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **1**, No. 3, pp. 367-382 (2009)

On Convergence of a Least-Squares Kansa's Method for the Modified Helmholtz Equations

Ting-On Kwok[1,*] and Leevan Ling[1]

[1] *Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong*

Received 19 December 2008; Accepted (in revised version) 12 March 2009
Available online 22 April 2009

Abstract. We analyze a least-squares asymmetric radial basis function collocation method for solving the modified Helmholtz equations. In the theoretical part, we proved the convergence of the proposed method providing that the collocation points are sufficiently dense. For numerical verification, direct solver and a subspace selection process for the trial space (the so-called adaptive greedy algorithm) is employed, respectively, for small and large scale problems.

AMS subject classifications: 35J25, 65N12, 65N15, 65N35

Key words: Radial basis function, adaptive greedy algorithm, asymmetric collocation, Kansa's method, convergence analysis.

1 Introduction

The unsymmetric RBF collocation method was first proposed by Kansa [6, 7]. Since then, many successful applications, from linear partial differential equations [17] to nonlinear shallow-water model [24], of recently developed mesh-free methods can be found in different Mathematics, Physics and Engineering journals.

From the theoretical point of view, the original unsymmetric RBF collocation formulation has neither error bounds nor convergence proofs. In the original formulation proposed by Kansa, the trial and test spaces were closely related; e.g. the set of collocation points and RBF centers coincide. This formulation may fail because the method results in singular systems in some specially constructed situations [5].

In order to carry out some mathematical analysis, it is necessary to make further assumptions and modify the formulation. In [11], we show that solvability can be guaranteed if the Kansa's method was modified in such a way that the test space and

*Corresponding author.
URL: `http://www.math.hkbu.edu.hk/~lling/`
Email: tokwok@math.hkbu.edu.hk (T.-O. Kwok), lling@hkbu.edu.hk (L. Ling)

trial space are de-linked. In particular, we show that sets of *proper* RBF centers exist so that the Kansa's resultant matrix is of full rank.

Later, we proposed in [13, 21] another variant of the method so that error bounds for the Poisson problems become possible. Convergence results and error bounds with respect to the $L^\infty(\Omega)$-norm are derived. A direct translation of theories to numerical algorithm results in solving an overdetermined resultant system with linear optimization whose implementation is not at all trivial; see [13] for an adaptive on-the-fly algorithm.

In [8], different formulations of the unsymmetric meshless collocation methods for solving the Poisson problems are compared in exact arithmetics. The numerical solution of convergent unsymmetric collocation method in [13] converges faster than the interpolant with respect to the residual norm. Most importantly, the numerical results in [8] suggests that, if the resulting overdetermined Kansa's system is solved by the least-squares minimization, the accuracy of the approximate PDE solution improves. This motivates the presented research.

In this paper, we are interested in the convergence theories of a radial basis function (RBF) method for solving the modified Helmholtz equation in strong form. In Section 2, we present the methodology of our proposed method. Section 3 devotes to the convergence proof of the proposed method that is done in three main parts. First, we give a brief overview of RBF *interpolation/approximation* theories. Next, a continuous dependency of the modified Helmholtz equation in a special form, which suits the least-squares approach in our formulation, is derived. Then, the denseness requirements of collocation points needed for our convergence results are studied. Finally, we put all the ideas together and show the convergence and error bounds. In Section 4, some numerical examples are given to conclude the work.

2 Overdetermined least-squares Kansa's method

Let $L := \Delta - k^2$, $k \in \mathbb{R}$, denote the modified Helmholtz operator and Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with boundary $\partial\Omega$. Moreover, suppose f is continuous in $\overline{\Omega}$ and g is continuous on $\partial\Omega$. We consider the modified Hemholtz equation with Dirichlet boundary conditions

$$Lu = f \quad \text{in } \Omega, \tag{2.1a}$$

$$u = g \quad \text{on } \partial\Omega. \tag{2.1b}$$

We assume that (2.1) has the exact solution u^* lying in some infinite dimensional trial spaces $\mathcal{U}$; we postpone the precise definition of $\mathcal{U}$ to Section 3.4 in which we prove the convergence of the proposed method.

To obtain a numerical formulation, we need to discretize $\mathcal{U}$ by some finite dimensional subspaces $\mathcal{U}_N$. The overdetermined least-squares based Kansa's method can be initialized by a user-defined set of N scattered RBF centers

$$\Xi_N := \{\xi_i\}_{i=1}^N.$$