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The Method of Fundamental Solutions for Solving Convection-Diffusion Equations with Variable Coefficients

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Abstract. A meshless method based on the method of fundamental solutions (MFS) is proposed to solve the time-dependent partial differential equations with variable coefficients. The proposed method combines the time discretization and the one-stage MFS for spatial discretization. In contrast to the traditional two-stage process, the one-stage MFS approach is capable of solving a broad spectrum of partial differential equations. The numerical implementation is simple since both closed-form approximate particular solution and fundamental solution are easy to find than the traditional approach. The numerical results show that the one-stage approach is robust and stable.

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Key words: Meshless method, method of fundamental solutions, particular solution, singular value decomposition, time-dependent partial differential equations.

1 Introduction

Through various types of reduction techniques, numerical solution of a given time-dependent partial differential equation can be obtained by converting it to a series of elliptic equations which can be solved by standard numerical methods. There are many reduction techniques which include Laplace transform method [4, 19], Fourier transform method [9], and discretization in time methods [7, 10, 11, 17]. Among these reduction techniques, the discretization in time methods appear to be the most popular approach. In this paper, we will focus on the method of discretization in time to

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reduce the given convection-diffusion problem to a series of elliptic partial differential equations.

For the purpose of solving time-dependent problems, instead of using traditional methods such as finite element, finite difference, or boundary element methods, we propose to apply the method of fundamental solutions (MFS) [2, 8, 10, 12] coupled with the method of particular solutions (MPS) with the use of radial basis functions. Such approach for solving time-dependent problems can be found in the literature [7, 10, 11, 17]. When the fundamental solution and particular solution of a given differential operator are available, the differential equation can be solved effectively. However, they can only be obtained for a limited class of linear differential operators. The fundamental solutions for various types of differential operators are available in the literature of boundary integral equations and boundary element methods (BEM). Furthermore, the closed-form particular solutions are available only for very limited classes of differential equations [5, 10, 14]. In the BEM, the dual reciprocity method (DRM) [15] has been successful in coupling the fundamental solution and particular solution to solve various science and engineering problems. However, for differential equations with variable coefficients, the above approach requires iterations and is not very effective. Recently, combining the MFS, MPS, and the DRM, it is possible to extend the above methods for solving elliptic partial differential equations with variable coefficients without the need of meshing the domain or boundary [3]. The idea of solving the given partial differential equation by combing the fundamental solution and particular solution as a one-stage method were proposed by Balakrishnan and Ramachandran [1] and Wang and Qin [18]. However, they seems unaware of the extended applications for solving PDEs with variable coefficients. The extensive study has been given and excellent results have been reported by Chen et al. [3] using the one-stage approach. Based on the numerical technique proposed in [3, 18], it is the purpose of this paper to extend the proposed one-stage method of the MFS and MPS to solve general convection-diffusion equations.

This paper is organized as follows. In Section 2, the θ-method has been applied to discretize the time domain. The given convection-diffusion equation is reduced to a series of elliptic differential equations. The MFS coupled with the MPS in the sense of one-stage formulation is applied to solve these elliptic equations at each time step. In Section 3, we conducted extensive numerical tests on two examples to demonstrate the convergence, stability, and high accuracy of the numerical algorithm mentioned in Section 2. In Section 4, we summarize the impact of each parameter to be used in the implementation.

2 Convection-diffusion equations

In this section, we consider the following general non-homogeneous time-dependent convection-diffusion equation in the closed domain $\Omega \subset \mathbb{R}^2$ bounded by $\partial\Omega$ given by

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = k\Delta u(\mathbf{x},t) + (\mathbf{v}\cdot\nabla)\, u(\mathbf{x},t) + f(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \tag{2.1}$$