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On the Boundary Integral Equations for a Two-Dimensional Slowly Rotating Highly Viscous Fluid Flow

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Abstract. In this paper, the two-dimensional slowly rotating highly viscous fluid flow in small cavities is modelled by the triharmonic equation for the streamfunction. The Dirichlet problem for this triharmonic equation is recast as a set of three boundary integral equations which however, do not have a unique solution for three exceptional geometries of the boundary curve surrounding the planar solution domain. This defect can be removed either by using modified fundamental solutions or by adding two supplementary boundary integral conditions which the solution of the boundary integral equations must satisfy. The analysis is further generalized to polyharmonic equations.

AMS subject classifications: 31A30, 31B10
Key words: Boundary integral equations, triharmonic and polyharmonic equations, logarithmic capacity.

1 Introduction

Mathematically, if one considers the incompressible rotating viscous flow [1], at large Ekman numbers, i.e., $E = \nu/(L^2\Omega) \gg 1$, which can be achieved if a highly viscous fluid with dynamic viscosity $\nu \gg 1$ is slowly rotating, i.e. the angular velocity $\Omega \ll 1$, in a small bounded cavity D of characteristic length $L \ll 1$, e.g., a square $[0, L] \times [0, L]$ lid-driven cavity [2], the model presented in [3] can be reduced to the triharmonic equation for the fluid streamfunction ψ, namely,

$$\nabla^6 \psi = 0, \quad \text{in } D \subset \mathbb{R}^2. \tag{1.1}$$

In this paper, we analyze the Dirichlet problem in which equation (1.1) has to be solved subject to the essential boundary conditions on the primary variables, namely,

$$\psi = f_0, \quad \frac{\partial \psi}{\partial n} = f_1, \quad \nabla^2 \psi = f_2, \quad \text{on } \partial D, \tag{1.2}$$

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where f_0, f_1 and f_2 are prescribed functions, ∂D is the boundary of the fluid domain D, and n is the outward unit normal to ∂D.

A direct boundary integral method for the interior Dirichlet problem for the two-dimensional Laplace equation, namely,

$$\nabla^2\theta = 0, \quad \text{in } D \subset \mathbb{R}^2, \quad \theta = \text{specified on } \partial D,$$

has been investigated in [4–7], whilst for the biharmonic equation, namely,

$$\nabla^4\phi = 0, \quad \text{in } D \subset \mathbb{R}^2, \qquad \phi \text{ and } \frac{\partial\phi}{\partial n} = \text{specified on } \partial D,$$

has been investigated in [8–10].

The purpose of this study is to extend these analyses to the triharmonic case given by equations (1.1) and (1.2), and make a classification for the polyharmonic equation

$$\nabla^{2k}\psi = 0, \quad \text{in } D \subset \mathbb{R}^2, \tag{1.3}$$

where $k \in \mathbb{N}^*$, which has to be solved subject to the boundary conditions

$$\left(\psi, \frac{\partial\psi}{\partial n}, \nabla^2\psi, \frac{\partial(\nabla^2\psi)}{\partial n}, ..., \frac{\partial(\nabla^{2p-2}\psi)}{\partial n}, \nabla^{2p}\psi\right) = \text{specified on } \partial D, \quad \text{if } k = 2p+1, \tag{1.4}$$

$$\left(\psi, \frac{\partial\psi}{\partial n}, \nabla^2\psi, \frac{\partial(\nabla^2\psi)}{\partial n}, ..., \nabla^{2p-2}\psi, \frac{\partial(\nabla^{2p-2}\psi)}{\partial n}\right) = \text{specified on } \partial D, \quad \text{if } k = 2p. \tag{1.5}$$

2 Boundary integral equations

We assume that the planar domain D is simply connected and bounded by a smooth, simple and closed contour ∂D, and that all the functions occurring in the sequel are as smooth as required by the process of mathematical manipulation in which they are involved.

Among the different methods which may be used for solving problem (1.1)-(1.2), the boundary element method (BEM) plays an important role. Here we are going to study a particular integral equation approach which emerges from the integral representation [3],

$$\begin{aligned}
&\int_{\partial D}\left[G_3(x,y)\frac{\partial(\nabla^4\psi)}{\partial n}(y) - \frac{\partial G_3}{\partial n(y)}(x,y)\nabla^4\psi(y)\right]ds(y)\\
&+\int_{\partial D}\left[\nabla_y^2 G_3(x,y)\frac{\partial(\nabla^2\psi)}{\partial n}(y) - \frac{\partial(\nabla_y^2 G_3)}{\partial n(y)}(x,y)\nabla^2\psi(y)\right]ds(y)\\
&+\int_{\partial D}\left[\nabla_y^4 G_3(x,y)\frac{\partial\psi}{\partial n}(y) - \frac{\partial(\nabla_y^4 G_3)}{\partial n(y)}(x,y)\psi(y)\right]ds(y)
\end{aligned}$$