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The Weak Galerkin Finite Element Method for Solving the Time-Dependent Integro-Differential Equations

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Abstract. In this paper, we solve linear parabolic integral differential equations using the weak Galerkin finite element method (WG) by adding a stabilizer. The semi-discrete and fully-discrete weak Galerkin finite element schemes are constructed. Optimal convergent orders of the solution of the WG in L^2 and H^1 norm are derived. Several computational results confirm the correctness and efficiency of the method.

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1 Introduction

Integro-differential equations [24] are used to simulate many phenomena in the fields of mathematics, dynamics and engineering technology [1, 9, 15]. It is also used in high-energy physics and biomedicine to help describe related physical phenomena and laws [7, 13]. Especially, in geology, the integro-differential equation [31] can be used to describe the fine three-dimensional problem of the interior of the earth to explore mineral products and predict earthquakes [8]. It also plays an important role in aerodynamics [3]. For example, it blue can be used to study the Brown displacement and thermal diffusion of suspended grain in heterogeneous fluid. When determining the profile of airfoil, the integro-differential equation can be used to calculate the effect of circulation, lift and

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resistance of the air [10]. Because of their application values, integro-differential equations are essential and significant research subjects. In this paper, we consider the linear parabolic integro-differential equation in domain $\Omega \subset \mathcal{R}^2$ with boundary $\partial\Omega$: seek an unknown function $u=u(\mathbf{x},t)$ satisfying:

$$u_t(\mathbf{x},t)-\nabla\cdot(A\nabla u(\mathbf{x},t))-\int_0^t \nabla\cdot(B\nabla u(\mathbf{x},\zeta))d\zeta=f(\mathbf{x},t), \qquad (\mathbf{x},t)\in\Omega\times(0,T], \tag{1.1a}$$

$$u=g(\mathbf{x},t), \qquad (\mathbf{x},t)\in\partial\Omega\times(0,T], \tag{1.1b}$$

$$u(\mathbf{x},0)=\psi(\mathbf{x}), \qquad \mathbf{x}\in\Omega, \tag{1.1c}$$

where $\mathbf{x}=\{x_1,x_2\}$, $A=[a_{i,j}(\mathbf{x},t)]_{2\times2}$, $B=[b_{i,j}(\mathbf{x},t)]_{2\times2}$, $A_t=[(a_{i,j})_t(\mathbf{x},t)]_{2\times2}$ and $B_t=[(b_{i,j})_t(\mathbf{x},t)]_{2\times2}$. The matrix-valued functions A and B are sufficiently smooth and A is symmetric. They also satisfy the some properties with positive constants α_1, α_2, α_3, β_1, β_2 for any $\xi,\eta\in\mathcal{R}^2$ that

$$\alpha_1\|\xi\|^2\le\xi^TA\xi\le\alpha_2\|\xi\|^2, \qquad \|\xi^TA_t\eta\|\le\alpha_3\|\xi\|\|\eta\|,$$
$$\|\xi^TB\eta\|\le\beta_1\|\xi\|\|\eta\|, \qquad \|\xi^TB_t\eta\|\le\beta_2\|\xi\|\|\eta\|.$$

Several numerical methods for problem (1.1a)-(1.1c) have been proposed. The earliest ones are the finite element (FE) methods [2, 4, 25] and the finite volume (FV) methods [14, 16]. One important characteristic of the finite element method is that it can preserve the conservation of mass and momentum. This method is primarily applied for the diffusion problems and the existence and uniqueness are proved. However, the finite volume method is preferred to the finite element method for conservation and stability. And it is more suitable for the discretization of the conservation of laws. A FV-FE method [12] which combines the advantages of the above methods is proposed. However, these methods require two mutually associated meshes. In order to reduce this correlation, many experts and scholars have proposed various discontinuous Galerkin methods. However, it is difficult to construct the penalty items of the discontinuous Galerkin method [5, 11].

Wang and Ye in 2011 proposed the weak Galerkin finite element for the second-order elliptic equations [19, 29]. The method is applied to many problems, such as Stokes equations [17, 18, 20–22, 28], Brinkman problem [23, 27], Biharmonic equations [30], eigenvalue problems [26] and Stochastic problems [6, 33] and so on. The partition of the domain can be arbitrary polygonal or polyhedral. The construction of the approximated function is simple and satisfies the stability condition. The essence of the weak Galerkin finite element is that the classical operators are replaced by some weak operators. In paper [32], the weak Galerkin finite element method is applied to the linear parabolic integro-differential equations. It proposes the semi-discrete and fully-discrete weak Galerkin finite element schemes. The optimal error estimates are obtained.

In this paper, we propose another weak finite element method by adding a stabilizer. The reason why we propose this method is that: firstly, the approximation space is easily constructed and simply satisfies the stability condition; moreover, the element of partition can be arbitrary polygon or polyhedron. That means, comparing with the literature [32], the weak Galerkin finite element method with a stabilizer is more effective in dealing with polygon meshes. In particular, polygonal meshes are widely used in the geological fields, such as earthquake prediction, coal mining and petroleum storage, etc.. Therefore, the weak Galerkin finite element method with a stabilizer is more flexible and has a wider range of applications. We also obtain the optimal order convergence in corresponding norms for the semi-discrete scheme and the fully-discrete scheme of the weak Galerkin finite element method, respectively.

The rest of this article is organized as follows. In Section 2, the preparatory work is presented. We propose semi-discrete and fully-discrete weak Galerkin finite element schemes in Section 3. The error estimates in H^1 norm and L^2 norm are derived in Section 4. In Section 5, we provide the weak Galerkin finite element method for the primal integro-differential. This part provides further theoretical support for the error estimates in Section 4. Finally, several experiments are presented to verify the validity of above theoretical analysis.

2 Preparatory work

In order to construct the variational forms of the weak Galerkin finite element method, we first introduce serval definitions and notations.

In this paper, we use the standard definitions in the Sobolev space $H^s(D)$. The associated inner products and norms are denoted by $(\cdot,\cdot)_{s,D}$ and $\|\cdot\|_{s,D}$ with any $s\geq 0$, respectively. The space $H^0(D)$ coincides with $L^2(D)$. When $D=\Omega$, we shall drop the subscripts D and s in the norm and inner product notation. When D is an edge/face, the L^2 inner product is represented by $\langle\cdot,\cdot\rangle_D$.

It is a well-known fact that the classical variational form of the parabolic integro-differential equations (1.1a)-(1.1c) is to find $u\in L^2(0,T;H^1(\Omega))$ satisfying respectively the initial and boundary conditions (1.1b) and (1.1c) with $t\in(0,T]$, such that

$$(u_t,v)+(A\nabla u,\nabla v)+\int_0^t(B\nabla u,\nabla v)d\zeta=(f,v),\qquad \forall v\in H_0^1(\Omega).$$

Assume that $\mathcal{T}_h$ is a partition of the domain Ω, which is a polygons in $\mathbb{R}^2$ or polyhedral in $\mathbb{R}^3$ and satisfies a set of shape regularity conditions [20]. Denote by T is an any element with ∂T as its boundary. Denote by $\mathcal{E}_h$ the set of all edges or faces in $\mathcal{T}_h$. For any element $T\in\mathcal{T}_h$, denote by h_T the diameter of T. Similarly, the diameter of $e\in\mathcal{E}_h$ is given by h_e. We define the mesh size of partition $\mathcal{T}_h$ as

$$h=\max_{T\in\mathcal{T}_h}h_T.$$

Next, we introduce the weak Galerkin finite element space

$$\begin{aligned}&V_h=\{v=\{v_0,v_b\},\ v_0\in P_k(T),\ v_b\in P_k(e),\ \forall T\in\mathcal{T}_h,\ \forall e\in\mathcal{E}_h\},\\&V_h^0=\{v\in V_h,\ v_b|_{\partial\Omega}=0\},\end{aligned}$$

where $P_k(T)$ and $P_k(e)$ are the sets of polynomials on T and e with the degree of polynomial no more than k. Note that, v_0 is the internal functions and v_b is the boundary functions on each element T. There is no relationship between v_0 and v_b. We emphasize that v_b has a single value on each edge $e\in\mathcal{E}_h$.

At last, we define the discrete weak gradient operator. For each $v=\{v_0,v_b\}\in V_h$, the discrete weak gradient [19] is denoted by $\nabla_{w,k-1}v$ and defined by the following equation

$$(\nabla_{w,k-1}v,\tau)=-(v_0,\nabla\cdot\tau)_T+\langle v_b,\tau\cdot n\rangle_{\partial T},\qquad \tau\in P_{k-1}(T).$$

In the following sections, we will drop the subscript $k-1$ of the discrete weak gradient with the confusion.

3 Weak Galerkin finite element method

With the partition of the domain Ω, we define some projections. On each element $T\in\mathcal{T}_h$, Q_0 is the L^2 projection from $L^2(T)$ onto $P_k(T)$. For each edge/face $e\in\mathcal{E}_h$, Q_b is the L^2 projection from $L^2(e)$ onto $P_k(e)$. Combing the projections Q_0 and Q_b are written to $Q_h=\{Q_0,Q_b\}$. Let R_h be a projection from $[L^2(T)]^d$ onto $[P_{k-1}(T)]^d$. The projections satisfy the commutative property $\nabla_w(Q_hu)=R_h(\nabla u)$, which is obtained by the definition of discrete weak gradient and integration by parts [20].

Introduce two bilinear forms with a matrix-valued function S in Ω

$$\begin{aligned} s(w,v)&=\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle w_0-w_b,v_0-v_b\rangle_{\partial T},\\ a(S,w,v)&=(S\nabla_w w,\nabla_w v)_T+s(w,v), \end{aligned}$$

for all $w,v\in V_h$.

From the above preparatory work, we present the semi-discrete weak Galerkin finite element schemes for the linear parabolic integro-differential equations.

Weak Galerkin Algorithm 1. Find $u_h(t)=\{u_0(t),u_b(t)\}\in L^2(0,T;V_h)$ satisfying

$$((u_0)_t,v_0)+a(A,u_h,v)+\int_0^t a(B,u_h,v)\,d\zeta=(f,v_0), \tag{3.1a}$$

$$u_b(t)=Q_bg, \tag{3.1b}$$

$$u_h(0)=E_hu(0), \tag{3.1c}$$

for any $v=\{v_0,v_b\}\in V_h^0$.

Let τ be the step size, $t_i=i\tau$ for $i=0,1,\cdots$, $u_h^i:=u_h(t_i)=\{u_0^i,u_b^i\}$ and $f^i:=f(t_i)$. At the time $t=t_i$, the backward Euler differential quotient is given by

$$\delta_tu_h^i=\frac{u_h^i-u_h^{i-1}}{\tau}.$$

With δ_t, we construct the fully-discrete weak Galerkin finite element scheme as follows

Weak Galerkin Algorithm 2. Find $u_h^i=\{u_0^i,u_b^i\}\in L^2(0,T;V_h)$ with any positive integer $1\le i\le N$ such that

$$(\delta_t u_0^i,v_0)+a(A^i,u_h^i,v)+\tau\sum_{j=0}^{i-1}a(B^j,u_h^j,v)=(f^i,v_0), \tag{3.2a}$$

$$u_b^i=Q_b g^i, \tag{3.2b}$$

$$u_h^0=E_h u^0, \tag{3.2c}$$

for any $v=\{v_0,v_b\}\in V_h^0$.

Define the norm of space V_h^0 as

$$|||v|||^2=a(I,v,v),$$

where I is an identity matrix.

Lemma 3.1. *$|||v|||$ is a norm of weak Galerkin finite element space V_h^0.*

Proof. When $|||v|||=0$, we get $\nabla_w v=0$ and $v_0=v_b$ on ∂T. From the fact $\nabla_w v=0$ and for any $q\in[P_{k-1}(T)]^d$, we have

$$\begin{aligned}0=&(\nabla_w v,q)_T\\ =&-(v_0,\nabla\cdot q)_T+\langle v_b,q\cdot\mathbf{n}\rangle_{\partial T}\\ =&(\nabla v_0,q)_T.\end{aligned}$$

Considering the arbitrariness of q and letting $q=\nabla v_0$, we obtain $\|\nabla v_0\|^2=0$. That yields $v_0=C$ on each element. It follows from $v_b=0$ on $\partial\Omega$ that $v_0=v_b=0$. □

Lemma 3.2. *For any $v,w\in V_h^0$, we have*

$$\begin{aligned}&|a(S,w,v)|\le C|||w|||\,|||v|||,\\ &|a(A,v,v)|\ge C|||v|||^2.\end{aligned}$$

Lemma 3.3. *For the numerical solution to the semi-discrete weak Galerkin finite element scheme* (3.1a) *with initial and boundary conditions* (3.1b) *and* (3.1c), *there holds that*

$$\|u_h(t)\|^2\le e^{Ct}\left(\|u_h(0)\|^2+C\int_0^t\|f(\zeta)\|^2\,d\zeta\right).$$

Proof. Letting $v=u_h$ in the form (3.1a), we have

$$((u_0)_t,u_0)+a(A,u_h,u_h)+\int_0^t a(B,u_h,u_h)\,d\zeta=(f,u_0).$$

It follows from bilinear property of $a(\cdot,\cdot,\cdot)$ that

$$((u_0)_t,u_0)\le(f,u_0),$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_0^2(t)\,dx=((u_0)_t,u_0)\le(f,u_0)\le C\left(\int_\Omega f^2\,dx+\int_\Omega u_0^2(t)\,dx\right).$$

Integrating the above inequality with respect to t and using Gronwall lemma, we complete the proof of this lemma. □

Theorem 3.1. *The solution of the semi-discrete weak Galerkin finite element schemes* (3.1a)-(3.1c) *is unique.*

Proof. It is enough to present that the following homogenous equations have a unique zero solution

$$((u_0)_t,v)+a(A,u_h,v)+\int_0^t a(B,u_h,v)\,d\zeta=0,\qquad \forall v\in V_h^0,\quad 0\le t\le T, \tag{3.3a}$$

$$u_b(t)=0\qquad \text{on }\partial\Omega,\quad 0\le t\le T, \tag{3.3b}$$

$$u_h(0)=\{0,0\}. \tag{3.3c}$$

Taking $v=u_h$ in (3.3a), we have

$$((u_0)_t,u_0)+a(A,u_h,u_h)+\int_0^t a(B,u_h,u_h)\,d\zeta=0.$$

Considering the positive definiteness of matrix A, the boundedness of the matrix B, the Cauchy-Schwarz inequality and the Young's inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(u_0,u_0)=\frac{1}{2}\frac{d}{dt}\int_\Omega u_0^2\,dx=((u_0)_t,u_0),\\&a(A,u_h,u_h)\ge\alpha_1|||u_h|||^2,\\&-\int_0^t a(B,u_h(\zeta),u_h(t))\,d\zeta\le C\int_0^t|||u_h(\zeta)|||^2\,d\zeta+\frac{\alpha_1}{2}|||u_h|||^2.\end{aligned}$$

It follows from these equations above that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_0^2\,dx+\frac{\alpha_1}{2}|||u_h|||^2\le C\int_0^t|||u_h(\zeta)|||^2\,d\zeta.$$

Integrating above equation from 0 to t, we obtain

$$\frac{1}{2}\|u_0(t)\|^2+\frac{\alpha_1}{2}\int_0^t|||u_h|||^2\,d\zeta\le C\int_0^t\int_0^\zeta|||u_h(\varsigma)|||^2\,d\varsigma\,d\zeta. \tag{3.4}$$

Since $\|u_0(t)\|^2 \geq 0$, so we have the fact

$$\frac{\alpha_1}{2}\int_0^t |||u_h|||^2 \, d\zeta \leq C\int_0^t \int_0^\zeta |||u_h(\varsigma)|||^2 \, d\varsigma \, d\zeta.$$

From Gronwall inequality, we get

$$\int_0^t |||u_h|||^2 \, d\zeta \leq 0.$$

Considering the property of norm $|||u_h|||\geq 0$, we obtain $u_h=0$, which means $u_0=u_b=0$. □

Theorem 3.2. *The solution of the fully-discrete weak Galerkin finite element schemes* (3.2a)-(3.2c) *is unique.*

Proof. It is analogous to prove the Theorem 3.1. Consider the following homogeneous equations

$$(\delta_t u_0^i, v_0) + a(A^i, u_h^i, v) + \tau \sum_{j=0}^{i-1} a(B^j, u_h^j, v) = 0, \qquad \forall v \in V_h^0, \quad 1 \leq i \leq N, \tag{3.5a}$$

$$u_b^i = 0 \qquad \text{on } \partial\Omega, \quad 1 \leq i \leq N, \tag{3.5b}$$

$$u_h^0 = \{0,0\}. \tag{3.5c}$$

Taking $v = u_h^i$ in (3.5a), we get

$$(\delta_t u_0^i, u_0^i) + a(A^i, u_h^i, u_h^i) + \tau \sum_{j=0}^{i-1} a(B^j, u_h^j, u_h^i) = 0, \qquad 1 \leq i \leq N.$$

It follows from backward Euler form, the positive definiteness and boundedness of $a(\cdot,\cdot,\cdot)$ and the Young's inequality that

$$a(A^i, u_h^i, u_h^i) \geq \alpha_1 |||u_h^i|||^2,$$
$$(\delta_t u_0^i, u_0^i) = \frac{1}{2\tau}\left((u_0^i, u_0^i) - (u_0^{i-1}, u_0^{i-1}) + (u_0^i - u_0^{i-1}, u_0^i - u_0^{i-1})\right) \geq \frac{1}{2\tau}(\|u_0^i\|^2 - \|u_0^{i-1}\|^2),$$

and

$$-\tau \sum_{j=0}^{i-1} a(B^j, u_h^j, u_h^i) \leq \beta_1 \tau \sum_{j=0}^{i-1} |||u_h^j||| \cdot |||u_h^i||| \leq C\beta_1 \tau \sum_{j=0}^{i-1} |||u_h^j|||^2 + \frac{\alpha_1}{2} |||u_h^i|||^2.$$

Summing up all above equations yields

$$\frac{1}{2\tau}(\|u_0^i\|^2 - \|u_0^{i-1}\|^2) + \frac{\alpha_1}{2} |||u_h^i|||^2 \leq C\beta_1 \tau \sum_{j=0}^{i-1} |||u_h^j|||^2. \tag{3.6}$$

Accumulating inequality (3.6) with i from 1 to $k-1$ for $1\le k\le N+1$, we obtain

$$\|u_0^{k-1}\|^2+\alpha_1\tau\sum_{i=1}^{k-1}|||u_h^i|||^2\le C\beta_1(\tau)^2\sum_{i=1}^{k-1}\sum_{j=1}^{i-1}|||u_h^j|||^2. \tag{3.7}$$

Considering the discrete Gronwall inequality with the fact $\|u_0^{k-1}\|^2\ge 0$, we arrive at

$$\tau\sum_{i=1}^{k-1}|||u_h^i|||^2\le 0, \qquad 1\le k\le N+1,$$

which yields $u_h^i=0$ with $1\le i\le N$. This theorem is completed. □

4 Error estimates

In this section, we derive the error equations and error estimates for the semi-discrete and fully-discrete weak Galerkin finite element schemes. First, we define an elliptical projection E_h, which is similar to Wheeler's projections in [23], for the exact solution $u\in L^2(0,T;H^1(\Omega))$ of the linear parabolic integro-differential equations (1.1a)-(1.1c). With E_h we define an equation with a fixed time $t\in(0,T]$ such that

$$\begin{aligned}&a(A,E_hu,v)+\int_0^t a(B,E_hu,v)\,d\zeta=(-\nabla\cdot(A\nabla u),v)-\int_0^t(\nabla\cdot B(\nabla u),v)\,d\zeta,\\ &E_bu=Q_bg \qquad \text{on } \partial\Omega,\end{aligned}$$

for any $v=\{v_0,v_b\}\in V_h^0$.

Denote

$$\begin{aligned}&e=E_hu-u_h, &&\rho=Q_hu-E_hu, &&\eta=u-Q_hu,\\ &\xi=Q_0u_t-\delta_t(Q_0u), &&\rho_t=Q_hu_t-E_hu_t, &&\eta_t=u_t-Q_hu_t.\end{aligned}$$

4.1 Semi-discrete WG error estimates

Firstly, we derive the semi-discrete error equations. Then, we use the error equations to derive the semi-discrete error estimates in H^1 norm and L^2 norm.

Lemma 4.1. *Assume $u_h\in L^2(0,T;V_h)$ is the numerical solution of semi-discrete WG schemes* (3.1a)-(3.1c) *and $u\in L^2(0,T;H^1(\Omega))$ is the exact solution of linear parabolic integro-differential equations* (1.1a)-(1.1c) *for $0<t\le T$. Then, for any $v\in V_h^0$, we have*

$$((e_0)_t,v_0)+a(A,e,v)+\int_0^t a(B,e,v)\,d\zeta=-((\rho_0)_t,v_0). \tag{4.1}$$

Proof. Testing the integral differential equation (1.1a) by $v=\{v_0,v_b\}\in V_h^0$ and considering the definition of projection E_h, we get

$$(u_t,v)+a(A,E_hu,v)+\int_0^t a(B,E_hu,v)\,d\zeta=(f,v_0). \tag{4.2}$$

It follows from the semi-discrete variational equation (3.1a) that

$$(E_0u_t-(u_0)_t,v_0)+a(A,E_hu-u_h,v)+\int_0^t a(B,E_hu-u,v)\,d\zeta=-(Q_0u_t-E_0u_t,v_0).$$

Following the definition of e and ρ, we obtain the semi-discrete error equation. $\Box$

Theorem 4.1. *Assume* $u\in L^2(0,T;H^{s+1}(\Omega))$ *and* $u_h\in L^2(0,T;V_h)$ *are the solutions of integro-differential equations* (1.1a)-(1.1c) *and semi-discrete weak Galerkin finite element variational forms* (3.1a)-(3.1c), *respectively. There exists a positive constant* C *independent of* h *satisfying*

$$\|e\|^2\le Ch^{2(s+1)}\int_0^t(\|u\|_{s+1}^2+\|u_t\|_{s+1}^2)\,d\zeta, \tag{4.3a}$$

$$|||e|||^2\le Ch^{2(s+1)}\int_0^t(\|u\|_{s+1}^2+\|u_t\|_{s+1}^2)\,d\zeta. \tag{4.3b}$$

Proof. Firstly, taking $v=e$ in the semi-discrete error equation (4.1), we have

$$((e_0)_t,e_0)+a(A,e,e)=-((\rho_0)_t,v_0)-\int_0^t a(B,e,e)\,d\zeta.$$

It follows from property of $a(\cdot,\cdot,\cdot)$, the Cauchy-Swarch inequality and the Young's inequality that

$$((e_0)_t,e_0)=\frac{1}{2}\frac{d}{dt}(e_0,e_0)=\frac{1}{2}\|e_0\|^2, \qquad a(A,e,e)\ge\alpha_1|||e|||^2,$$
$$-\int_0^t a(B,e,e)\le\int_0^t\beta_1|||e|||^2\,d\zeta, \qquad -((\rho_0)_t,e_0)\le C\|(\rho_0)_t\|^2+\frac{1}{2}\|e_0\|^2,$$

which leads to

$$\frac{d}{dt}\|e_0\|^2+|||e|||^2\le C\int_0^t|||e|||^2\,d\zeta+C(\|(\rho_0)_t\|^2+\|e_0\|^2).$$

Integrating above equation from 0 to t with the fact $e(\cdot,0)=0$ yields

$$\|e_0\|^2+\int_0^t|||e|||^2\,d\zeta\le C\int_0^t\left(\int_0^\tau|||e|||^2\,d\varsigma+\|e_0\|^2\right)d\zeta+C\int_0^t\|(\rho_0)_t\|^2\,d\zeta.$$

By the Gronwall's inequality and the estimate of $\|(\rho_0)_t\|$ in the Theorem 5.3, we obtain an error estimate formula (4.3a).

Next, for each fixed t, letting $v=e_t$ in semi-discrete error equation (4.1) yields

$$((e_0)_t,(e_0)_t)+a(A,e,e_t)=-\int_0^t a(B,e,e_t)\,d\zeta-((\rho_0)_t,(e_0)_t).$$

Since $\frac{d}{dt}a(S,e,e)=a(S_t,e,e)+2a(S,e,e_t)$ with $S\in\{A,B\}$, we obtain

$$\begin{aligned}&\|(e_0)_t\|^2+\frac{1}{2}\frac{d}{dt}a(A,e,e)\\=&\frac{1}{2}a(A_t,e,e)-\frac{1}{2}a(B,e,e)+\frac{1}{2}\int_0^t a(B_t,e,e)\,d\zeta-((\rho_0)_t,(e_0)_t).\end{aligned}\tag{4.4}$$

Using the definition of $|||\cdot|||$, the property of A and B and the Young's inequality, we obtain

$$\begin{aligned}&a(A,e,e)\geq\alpha_1|||e|||^2, && a(A_t,e,e)\leq\alpha_3|||e|||^2,\\ &-a(B,e,e)\leq\beta_1|||e|||^2, && a(B_t,e,e)\leq\beta_2|||e|||^2,\end{aligned}$$

and

$$-((\rho_0)_t,(e_0)_t)\leq\frac{1}{4}\|(\rho_0)_t\|^2+\|(e_0)_t\|^2.$$

Substituting all these forms into (4.4) yields

$$\frac{1}{2}\frac{d}{dt}|||e|||^2\leq C(|||e|||^2+\|(\rho_0)_t\|^2)+C\int_0^t|||e|||^2\,d\zeta.$$

Integrating the above formula with respect to t, we have

$$|||e|||^2\leq C\int_0^t|||e|||^2\,d\zeta+C\int_0^t\|(\rho_0)_t\|^2\,d\zeta.$$

From Gronwall inequality and the estimate of $\|(\rho_0)_t\|$, we obtain an error estimate formula (4.3b). □

4.2 Fully-discrete WG error estimates

Firstly, we derive the fully discrete error equations. Then, we use the error equations to derive the fully-discrete error estimates in H^1 norm and L^2 norm.

Lemma 4.2. *Let $u_h\in L^2(0,T;V_h)$ be the numerical solution of fully-discrete WG schemes* (3.2a)-(3.2c) *and $u\in L^2(0,T;H^1(\Omega))$ be the exact solution of linear parabolic integro-differential equations* (1.1a)-*1.1c with $0<t\leq T$. Then, for any $v\in V_h^0$ and $1\leq i\leq N$, we have*

$$(\delta_t e_0^i,v_0)+a(A^i,e^i,v)=-(\xi_0^i,v_0)-(\delta_t\rho_0^i,v_0)-\int_0^{t^i}a(B,E_hu,v)d\zeta+\tau\sum_{j=0}^{i-1}a(B^j,u_h^j,v).$$

Proof. Using $v=\{v_0,v_b\}\in V_h^0$ to test the integro-differential equation (1.1a) and the definition of E_h, we have

$$(Q_0u_t,v_0)+a(A,E_hu,v)+\int_0^t a(B,E_hu,v)\,d\zeta=(f,v_0).$$

Considering the fully-discrete variational forms and the equation above with $t=t^i$ and $1\le i\le N$, we obtain

$$(\delta_t e_0^i,v_0)+a(A^i,e^i,v)=-(\xi_0^i,v_0)-(\delta_t\rho_0^i,v_0)-\int_0^{t^i} a(B,E_hu,v)\,d\zeta+\tau\sum_{j=0}^{i-1}a(B^j,u_h^j,v).$$

Thus, we complete the proof. □

Similarly, following the error estimates of semi-discrete WG schemes, we construct the error estimates of fully-discrete WG schemes in H^1 norm and L^2 norm, respectively.

Theorem 4.2. *Let $u_h^n\in L^2(0,T;V_h)$ be the solution of the problems* (5.3a)-(5.3b) *arising from fully-discrete weak Galerkin finite element schemes* (3.2a)-(3.2c). *Assume $u\in L^2(0,T;H^{s+1}(\Omega))$ is the exact solution of integro-differential problems* (5.3a)-(5.3b). *Then, for any $v\in V_h^0$, $1\le k\le N$ and $n=0,1\cdots$, we obtain*

$$\begin{aligned}\|e_0^k\|^2\le & C\tau^2\left(\|u_{tt}\|^2_{L^2(0,T;L^2)}+\|u\|^2_{L^2(0,T;H^{s+1})}+\|u_t\|^2_{L^2(0,T;H^{s+1})}\right)\\ &+Ch^{2(s+1)}\left(\|u\|^2_{L^2(0,T;H^{s+1})}+\|u_t\|^2_{L^2(0,T;H^{s+1})}\right).\end{aligned}$$

Proof. Taking $v=e^i$ in Lemma 4.2 for the fully-discrete error equation, we have

$$\begin{aligned}&(\delta_t e_0^i,e_0^i)+a(A^i,e^i,e^i)\\ =&-(\eta_0^i,e_0^i)-(\delta_t\rho_0^i,e_0^i)-\int_0^{t^i}a(B,E_hu,e^i)\,d\zeta+\tau\sum_{j=0}^{i-1}a(B^j,E_hu(t^j),e^i)\\ &-\tau\sum_{j=0}^{i-1}a(B^j,E_hu(t_j)-u_h^j,e^i),\qquad 1\le i\le N.\end{aligned}$$

It follows from the backward Euler form, the definition of $|||\cdot|||$ and the Young's inequality that

$$\begin{aligned}&|(\delta_t e_0^i,e_0^i)|=\frac{1}{2\tau}((e_0^i,e_0^i)-(e_0^{i-1},e_0^{i-1})+(e_0^i-e_0^{i-1},e_0^i-e_0^{i-1}))\ge\frac{1}{2\tau}(\|e_0^i\|^2-\|e_0^{i-1}\|^2),\\ &|a(A^i,e^i,e^i)|\ge\alpha_1|||e^i|||^2.\end{aligned}$$

Notice that

$$\xi_0^i=Q_0u_t^i-\delta_t(Q_0u^i)=Q_0(u_t^i-\delta_tu^i)=Q_0\left(\frac{1}{\tau}\int_{t^{i-1}}^{t^i}(\zeta-t^{i-1})u_{tt}(\zeta)\,d\zeta\right),$$

we obtain

$$\|\xi_0^i\| \leq C\| \int_{t^{i-1}}^{t^i} |u_{tt}(\zeta)| d\zeta \| \leq C \left(\tau \int_{t^{i-1}}^{t^i} \|u_{tt}\|^2 d\zeta \right)^{\frac{1}{2}}.$$

So, we have

$$|(\xi_0^i, e_0^i)| \leq C\|\xi_0^i\|^2 + \frac{1}{4}\|e_0^i\|^2 \leq C\tau \int_{t^{i-1}}^{t^i} \|u_{tt}\|^2 \, d\zeta + \frac{1}{4}\|e_0^i\|^2.$$

Thanks to

$$\|\delta_t \rho_0^i\|^2 = \left\| \frac{\rho_0^i - \rho_0^{i-1}}{\tau} \right\|^2 = \frac{1}{\tau^2} \left\| \int_{t^{i-1}}^{t^i} (\rho_0)_t \, d\zeta \right\|^2 \leq \frac{1}{\tau} \int_{t^{i-1}}^{t^i} \|(\rho_0)_t\|^2 \, d\zeta,$$

we get

$$|-(\delta_t \rho_0^i, e_0^i)| \leq \frac{C}{\tau} \int_{t^{i-1}}^{t^i} \|(\rho_0)_t\|^2 \, d\zeta + \frac{1}{4}\|e_0^i\|^2.$$

Since

$$\frac{d}{dt}(B\nabla_w(E_h u)) = B_t \nabla_w(E_h u) + B\nabla_w(E_h u_t),$$

we obtain

$$\begin{aligned}
&\left| \int_0^{t^i} a(B, E_h u, e^i) \, d\zeta - \tau \sum_{j=0}^{i-1} a(B^j, E_h u(t^j), e^i) \right| \\
= &\int_0^{t^i} s(E_h u, e^i) \, d\zeta - \tau \sum_{j=0}^{i-1} s(E_h u(t^j), e^i) \\
&\quad + \left(\sum_{j=0}^{i-1} \int_{t^j}^{t^{j+1}} [B\nabla_w(E_h u(\zeta)) - B^j \nabla_w(E_h u(t^j))] \, d\zeta, \nabla_w e^i \right).
\end{aligned}$$

We have the fact

$$\begin{aligned}
&\left| \left(\sum_{j=0}^{i-1} \int_{t^j}^{t^{j+1}} [B\nabla_w(E_h u(\zeta)) - B^j \nabla_w(E_h u(t^j))] \, d\zeta, \nabla_w e^i \right) \right| \\
= &\left(\sum_{j=0}^{i-1} \int_{t^j}^{t^{j+1}} \int_{t^j}^{\zeta} \frac{d}{d\varsigma} [B\nabla_w(E_h u(\varsigma))] \, d\varsigma d\zeta, \nabla_w e^i \right) \\
= &\left(\sum_{j=0}^{i-1} \int_{t^j}^{t^{j+1}} \int_{t^j}^{\zeta} [B_t \nabla_w(E_h(\varsigma)) + B\nabla_w(E_h u_t(\varsigma))] \, d\varsigma d\zeta, \nabla_w e^i \right) \\
\leq &\left(\sum_{j=0}^{i-1} \Delta t \int_{t^j}^{t^{j+1}} [B_t \nabla_w(E_h(\varsigma)) + B\nabla_w(E_h u_t(\varsigma))] d\varsigma, \nabla_w e^i \right)
\end{aligned}$$

$$
\begin{aligned}
&=\left(\tau\int_0^{t^i}[B_t\nabla_w(E_h(\zeta))+B\nabla_w(E_hu_t(\zeta))]\,d\zeta,\nabla_w e^i\right)\\
&\le C\tau^2\left(\|u\|^2_{L^\infty(0,T;H^{s+1})}+\|u_t\|^2_{L^\infty(0,T;H^{s+1})}\right)+\frac{\alpha_1}{6}|||e^i|||^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_0^{t^i}s(E_hu,e^i)\,d\zeta-\tau\sum_{j=0}^{i-1}s(E_hu(t^j),e^i)\right|\\
=&\sum_{j=0}^{i-1}\int_{t^j}^{t^{j+1}}s(E_hu,e^i)\,d\zeta-\sum_{j=0}^{i-1}\int_{t^j}^{t^{j+1}}s(E_hu(t^j),e^i)\,d\zeta\\
=&\sum_{j=0}^{i-1}\int_{t^j}^{t^{j+1}}s(E_hu-E_hu(t^j),e^i)\,d\zeta\\
=&\sum_{j=0}^{i-1}\int_{t^j}^{t^{j+1}}\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle(E_0u-E_bu)-(E_0u(t^j)-E_bu(t^j)),e_0^i-e_b^i\rangle_{\partial T}\,d\zeta\\
\le&\sum_{T\in\mathcal{T}_h}\left\langle\sum_{j=0}^{i-1}\tau\int_{t^j}^{t^{j+1}}h_T^{-1}[E_0u_t(\zeta)-E_bu_t(\zeta)]\,d\zeta,e_0^i-e_b^i\right\rangle_{\partial T}\\
\le&\sum_{T\in\mathcal{T}_h}\tau\left\langle\int_0^{t^i}h_T^{-1}[E_0u_t(\zeta)-E_bu_t(\zeta)]\,d\zeta,e_0^i-e_b^i\right\rangle_{\partial T}\\
\le&C\tau^2\|u_t\|^2_{L^\infty(0,T;H^{s+1})}+\frac{\alpha_1}{6}|||e^i|||^2.
\end{aligned}
$$

From the property of $a(\cdot,\cdot,\cdot)$, we get

$$
\begin{aligned}
&\tau\sum_{j=0}^{i-1}a(B^j,E_hu(t^j)-u_h^j,e^i)\le\beta_1\tau\sum_{j=0}^{i-1}|||E_hu(t^j)-u_h^j|||\,|||e^i|||\\
\le&C\tau^2\sum_{j=0}^{j-1}|||e^j|||^2+\frac{\alpha_1}{6}|||e^i|||^2.
\end{aligned}
$$

Combing all the equations above, we get

$$
\begin{aligned}
&\|e_0^i\|^2-\|e_0^{i-1}\|^2+2\alpha_1\tau|||e^i|||^2\\
\le&C\tau^2\int_{t^{i-1}}^{t^i}\|u_{tt}\|^2\,d\zeta+\tau\|e_0^i\|^2+C\int_{t^{i-1}}^{t^i}\|(\rho_0)_t\|^2d\zeta\\
&+C\zeta^3\left(\|u\|^2_{L^\infty(0,T;H^{s+1})}+\|u_t\|^2_{L^\infty(0,T;H^{s+1})}\right)+\alpha_1\tau|||e^i|||^2+C\tau^3\sum_{j=0}^{j-1}|||e^j|||^2.
\end{aligned}
$$

Accumulating the above equation with i from 1 to $k-1$, we obtain

$$\begin{aligned}&\|e_0^{k-1}\|^2+\alpha_1\tau\sum_{i=1}^{k-1}|||e|||^2\\ \leq& C\tau^2\int_0^{t^{k-1}}\|u_{tt}\|^2\,d\zeta+\tau\sum_{i=1}^{k-1}\|e_0^i\|^2+C\int_0^{t^k}\|(\rho_0)_t\|^2\,d\zeta\\ &+C\tau^2\left(\|u\|^2_{L^\infty(0,T;H^{s+1})}+\|u_t\|^2_{L^\infty(0,T;H^{s+1})}\right)+C\tau^3\sum_{i=1}^{k-1}\sum_{j=1}^{i-1}|||e^j|||^2,\end{aligned}$$

where $1\leq k\leq N+1$. Following the Gronwall inequality and the estimates of $|||\rho|||$, we obtain the fully-discrete error estimate. □

Theorem 4.3. *Let $u_h^n\in V_h$ be the solution of the problems* (5.3a)-(5.3b) *arising from fully-discrete weak Galerkin finite element schemes* (3.2a)-(3.2c). *Assume $u\in L^2(0,T;H^{s+1}(\Omega))$ is the exact solution of* (5.3a)-(5.3b). *Then, for any $v\in V_h$, $1\leq k\leq N$ and $n=0,1\cdots$, we have*

$$\begin{aligned}|||e^k|||^2\leq& C\tau^2\left(\|u_{tt}\|^2_{L^2(0,T;H^{s+1})}+\|u_t\|^2_{L^\infty(0,T;L^2)}+\|u\|^2_{L^\infty(0,T;H^{s+1})}\right)\\ &+Ch^{2(s+1)}\left(\|u_t\|^2_{L^2(0,T;H^{s+1})}+\|u\|^2_{L^2(0,T;H^{s+1})}\right).\end{aligned}$$

Proof. Letting $v=\delta_te^i$ in the fully-discrete error equation in Lemma 4.2, we obtain

$$\begin{aligned}&\|\delta_te_0^i\|^2+a(A^i,e^i,\delta_te^i)\\ =&-(\xi_0^i,\delta_te_0^i)-(\delta_t\rho_0^i,\delta_te_0^i)-\int_0^{t^i}a(B,E_hu,\delta_te^i)\,d\zeta+\tau\sum_{i_0=0}^{i-1}a(B^{i_0},u_h^{i_0},\delta_te^i)\\ =&-(\xi_0^i,\delta_te_0^i)-(\delta_t\rho_0^i,\delta_te_0^i)+\delta_t\left(\int_0^{t^i}a(B,E_hu,e^i)\,d\zeta-\tau\sum_{j=0}^{i-1}a(B^j,u_h^j,e^i)\right)\\ &-\frac{1}{\tau}\left(\int_{t^{i-1}}^{t^i}a(B,E_hu,e^{i-1})\,d\zeta-\tau a(B^{i-1},E_hu(t^{i-1}),e^{i-1})\right)-a(B^{i-1},e^{i-1},e^{i-1}).\end{aligned}$$

We estimate the second term and the right hand side terms as follows

$$\begin{aligned}&a(A^i,e^i,\delta_te^i)\geq\frac{1}{2\tau}(a(A^i,e^i,e^i)-a(A^i,e^i,e^{i-1})),\\ &-(\xi_0^i,\delta_te_0^i)|\leq C(\|\xi_0^i\|^2+\frac{1}{4}\|\delta_te_0^i\|^2),\\ &|-(\delta_t\rho_0^i,\delta_te_0^i)|\leq C\|\delta_t\rho_0^i\|^2+\frac{1}{4}\|\delta_te_0^i\|^2,\\ &|a(B^{i-1},e^{i-1},e^{i-1})|\leq C|||e^{i-1}|||^2.\end{aligned}$$

Notice that

$$\begin{aligned}&\left|\frac{1}{\tau}\left(\int_{t^{i-1}}^{t^i} a(B,E_hu,e^{i-1})\,d\zeta-\tau a(B^{i-1},E_hu(t^{i-1}),e^{i-1})\right)\right|\\=&\left|\frac{1}{\tau}\left(\int_{t^{i-1}}^{t^i}(B\nabla_w(E_hu),\nabla_we^{i-1})\,d\zeta-\tau(B^{i-1}\nabla_w(E_hu(t^{i-1})),\nabla_we^{i-1})\right)\right|\\&\quad+\left|\frac{1}{\tau}\left(\int_{t^{i-1}}^{t^i}s(E_hu,e^{i-1})\,d\zeta-\tau s(E_hu,e^{i-1})\right)\right|,\end{aligned}$$

where

$$\begin{aligned}&\left|\frac{1}{\tau}\left(\int_{t^{i-1}}^{t^i}(B\nabla_w(E_hu),\nabla_we^{i-1})\,d\zeta-\tau(B^{i-1}\nabla_w(E_hu(t^{i-1})),\nabla_we^{i-1})\right)\right|\\=&\left|\left(\frac{1}{\tau}\int_{t^{i-1}}^{t^i}(B\nabla_w(E_hu)-B^{i-1}\nabla_w(E_hu(t^{i-1})))\,d\zeta,\nabla_we^{i-1}\right)\right|\\=&\left|\left(\frac{1}{\tau}\int_{t^{i-1}}^{t^i}\int_{t^{i-1}}^{\zeta}\frac{d}{d\varsigma}(B\nabla_w(E_hu(\varsigma)))\,d\varsigma d\zeta,\nabla_we^{i-1}\right)\right|\\=&\left|\left(\frac{1}{\tau}\int_{t^{i-1}}^{t^i}\int_{t^{i-1}}^{\zeta}(B_t\nabla_w(E_hu)+B\nabla_w(E_hu_t))d\varsigma d\zeta,\nabla_we^{i-1}\right)\right|\\\le& C\tau\left(\beta_1\|u\|^2_{L^\infty(0,T;H^{s+1})}+\beta_2\|u_t\|^2_{L^\infty(0,T;H^{s+1})}\right)^{\frac{1}{2}}|||e^{i-1}|||\\\le& C\tau^2\left(\beta_1\|u\|^2_{L^\infty(0,T;H^{s+1})}+\beta_2\|u_t\|^2_{L^\infty(0,T;H^{s+1})}\right)+|||e^{i-1}|||^2,\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{\tau}\left(\int_{t^{i-1}}^{t^i}s(E_hu,e^{i-1})\,d\zeta-\tau s(E_hu,e^{i-1})\right)\right|\\=&\left|\frac{1}{\tau}\int_{t^{i-1}}^{t^i}s(E_hu-E_hu(t^{i-1}),e^{i-1})\,d\zeta\right|\\=&\left|\frac{1}{\tau}\int_{t^{i-1}}^{t^i}\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle E_0u-E_bu-(E_0u(t^{i-1})-E_bu(t^{i-1})),e_0^{i-1}-e_b^{i-1}\rangle_{\partial T}\,d\zeta\right|\\\le&\left|\sum_{T\in\mathcal{T}_h}\left\langle\int_{t^{i-1}}^{t^i}h_T^{-1}(E_0u_t(\zeta)-E_bu_t(\zeta))\,d\zeta,e_0^{i-1}-e_b^{i-1}\right\rangle_{\partial T}\right|\\\le& C\tau^2\|u\|^2_{L^\infty(0,T;H^{s+1})}+|||e^{i-1}|||^2.\end{aligned}$$

Summing the equations above, we get

$$\begin{aligned}&\tau\|\delta_t e_0^i\|^2+a(A^i,e^i,e^i)-a(A^i,e^i,e^{i-1})\\ \le& C\tau^2\int_{t^{i-1}}^{t^i}\|u_{tt}\|^2\,d\zeta+C\int_{t^{i-1}}^{t^i}\|(\rho_0)_t\|^2\,d\zeta+C\tau|||e^{i-1}|||^2\\ &+C\tau^3\Big(\beta_1\|u\|^2_{L^\infty(0,T;H^{s+1})}+\beta_2\|u_t\|^2_{L^\infty(0,T;H^{s+1})}\Big)\\ &+2\tau\delta_t\left(\int_0^{t^i}a(B,E_hu,e^i)\,d\zeta-\tau\sum_{j=0}^{i-1}a(B^j,u_h^j,\delta_t e^i)\right).\end{aligned}$$

Adding the above with respect to i from 1 to k with $e_0^0=0$ and $1\le k\le N$, we have

$$\begin{aligned}|||e^k|||^2\le& C\tau^2\int_0^{t^k}\|u_{tt}\|^2\,d\zeta+C\int_0^{t^k}\|(\rho_0)_t\|^2\,d\zeta+C\tau\sum_{i=1}^{k}|||e^i|||^2\\ &+C\tau^2\Big(\|u\|^2_{L^\infty(0,T;H^{s+1})}+\|u_t\|^2_{L^\infty(0,T;H^{s+1})}\Big)\\ &-2\left(\int_0^{t^k}a(B,E_hu,e^k)\,d\zeta-\tau\sum_{j=0}^{k-1}a(B^j,u_h^j,e^k)\right).\end{aligned}$$

It follows from the proof of the estimates of $\|e_0^k\|$ that

$$\begin{aligned}&\int_0^{t^k}a(B,E_hu,e^k)\,d\zeta-\tau\sum_{j=0}^{k-1}a(B^j,u_h^j,e^k)\\ \le& C\tau^2\Big(\|u\|^2_{L^\infty(0,T;H^{s+1})}+\|u_t\|^2_{L^\infty(0,T;H^{s+1})}\Big)+\frac{1}{2}|||e^k|||^2.\end{aligned}$$

Using the Gronwall inequality and the estimate of $\|(\rho_0)_t\|$, we have

$$\begin{aligned}|||e^k|||^2\le& C\tau^2\Big(\|u_{tt}\|^2_{L^2(0,T;H^{s+1})}+\|u_t\|^2_{L^\infty(0,T;L^2)}+\|u\|^2_{L^\infty(0,T;H^{s+1})}\Big)\\ &+Ch^{2(s+1)}\Big(\|u_t\|^2_{L^2(0,T;H^{s+1})}+\|u\|^2_{L^2(0,T;H^{s+1})}\Big).\end{aligned}$$

Thus, we complete the proof. □

5 WG for primary integro-differential equation

In this section, we firstly present several results regarding approximation properties of the L^2 projections R_h and Q_h. Then, we provide the H^1 norm and L^2 norm for the linear integro-differential equation without the item u_t, respectively, to support the analysis of previous error estimates.

Lemma 5.1 ([19])**.** *Let $\mathcal{T}_h$ be a finite element partition of Ω satisfying the shape regularity* [20] *and $w\in H^{r+1}$ and $\rho\in H^r(\Omega)$ with $1\le r\le k$. Then, for $0\le s\le 1$ we have*

$$\sum_{T\in\mathcal{T}_h} h_T^{2s}\|w-Q_0w\|_{T,s}^2\le Ch^{2(r+1)}\|w\|_{r+1}^2,$$
$$\sum_{T\in\mathcal{T}_h} h_T^{2s}\|\nabla w-Q_0(\nabla w)\|_{T,s}^2\le Ch^{2r}\|w\|_{r+1}^2.$$

Theorem 5.1. *Let $u\in H^{s+1}(\Omega)$ be the exact solution of linear integro-differential equation without the item u_t. According to the definition of ρ, then*

$$|||\rho|||\le Ch^s\left(\|u\|_{s+1}^2+\int_0^t\|u\|_{s+1}^2\,d\zeta\right)^{\frac{1}{2}},\qquad 0\le t\le T. \tag{5.1}$$

Proof. It follows from the definition of E_h and integrating by parts with any $v\in V_h^0$ that

$$\begin{aligned}
&a(A,E_hu,v)+\int_0^t a(B,E_hu,v)\,d\zeta\\
=&-(\nabla\cdot(A\nabla u),v_0)-\int_0^t(\nabla\cdot(B\nabla u),v_0)\,d\zeta\\
=&(A\nabla u,\nabla v_0)-\sum_{T\in\mathcal{T}_h}\langle A\nabla u\cdot\mathbf{n},v_0-v_b\rangle_{\partial T}+\int_0^t(B\nabla u,\nabla v_0)\,d\zeta\\
&-\int_0^t\sum_{T\in\mathcal{T}_h}\langle B\nabla u\cdot\mathbf{n},v_0-v_b\rangle_{\partial T}\,d\zeta.
\end{aligned}$$

By the definition of $a(\cdot,\cdot,\cdot)$ and discrete weak gradient, commutativity and integrating by parts, we have

$$\begin{aligned}
&a(A,Q_hu,v)+\int_0^t a(B,Q_hu,v)\,d\zeta\\
=&(AR_h(\nabla u),\nabla_w v)+\int_0^t(BR_h(\nabla u),\nabla_w v)\,d\zeta+s(Q_hu,v)+\int_0^t s(Q_hu,v)\,d\zeta\\
=&(AR_h(\nabla u),\nabla v_0)+\sum_{T\in\mathcal{T}_h}\langle AR_h(\nabla u)\cdot\mathbf{n},v_b-v_0\rangle_{\partial T}+s(Q_hu,v)+\int_0^t s(Q_hu,v)\,d\zeta\\
&+\int_0^t(BR_h(\nabla u),\nabla v_0)\,d\zeta+\int_0^t\sum_{T\in\mathcal{T}_h}\langle BR_h(\nabla u)\cdot\mathbf{n},v_b-v_0\rangle_{\partial T}\,d\zeta.
\end{aligned}$$

From the above two equations, we obtain

$$\begin{aligned}
&a(A,\rho,v)+\int_0^t a(B,\rho,v)\,d\zeta\\
=&(A(R_h(\nabla u)-\nabla u),\nabla v_0)+\int_0^t(B(R_h(\nabla u)-\nabla u),\nabla v_0)\,d\zeta\\
&+l(A,u,v)+\int_0^t l(B,u,v)\,d\zeta+s(Q_hu,v)+\int_0^t s(Q_hu,v)\,d\zeta,
\end{aligned}\tag{5.2}$$

where

$$l(S,u,v)=\sum_{T\in\mathcal{T}_h}\langle S(R_h(\nabla u)-\nabla u)\cdot\mathbf{n},v_b-v_0\rangle_{\partial T}.$$

Taking $v=\rho$ in the error equation (5.2), we get

$$\begin{aligned}a(A,\rho,\rho)=&-\int_0^t a(B,\rho,\rho)\,d\zeta+(A(R_h(\nabla u)-\nabla u),\nabla\rho_0)+\int_0^t l(B,u,\rho)\,d\zeta\\&+l(A,u,\rho)+s(Q_hu,\rho)+\int_0^t(B(R_h(\nabla u)-\nabla u),\nabla\rho_0)\,d\zeta\\&+\int_0^t s(Q_hu,\rho)\,d\zeta.\end{aligned}$$

The properties of A and B, the Cauchy-Schwarz, approximation properties and the trace inequality yield

$$\begin{aligned}&\|a(A,\rho,\rho)\|\geq\alpha_1|||\rho|||^2,\\&|l(A,u,\rho)|=\Big|\sum_{T\in\mathcal{T}_h}\langle A(R_h(\nabla u)-\nabla u)\cdot\mathbf{n},\rho_0-\rho_b\rangle_{\partial T}\Big|\\&\qquad\leq C|||\rho|||\cdot\left(\sum_{T\in\mathcal{T}_h}\|R_h(\nabla u)-u\|_T^2+h^2\|\nabla(R_h(\nabla u)-u)\|_T^2\right)^{1/2}\\&\qquad\leq Ch^{2s}\|u\|_{s+1}^2+\frac{\alpha_1}{6}|||\rho|||^2,\\&|s(Q_hu,\rho)|=\left|\sum_{T\in\mathcal{T}_h}h^{-1}\langle Q_0u-Q_bu,\rho_0-\rho_b\rangle_{\partial T}\right|\\&\qquad\leq\left|\sum_{T\in\mathcal{T}_h}h^{-1}\langle Q_0u-u,\rho_0-\rho_b\rangle_{\partial T}\right|\\&\qquad\leq Ch^{2s}\|u\|_{s+1}^2+\frac{\alpha_1}{6}|||\rho|||^2,\end{aligned}$$

and

$$\begin{aligned}&\|(A(R_h(\nabla u)-\nabla u),\nabla\rho_0)\|\leq C\|R_h(\nabla u)-\nabla u\|\cdot\|\nabla\rho_0\|\\&\qquad\leq Ch^{2s}\|u\|_{s+1}^2+\frac{\alpha_1}{6}|||\rho|||^2,\\&\left|\int_0^t(B(R_h(\nabla u)-\nabla u),\nabla\rho_0)\,d\zeta\right|\leq Ch^{2s}\int_0^t\|u\|_{s+1}^2\,d\zeta+C\int_0^t|||\rho|||^2\,d\zeta,\\&\left|-\int_0^t a(B,\rho,\rho)\,d\zeta\right|\leq\beta_1\int_0^t|||\rho|||^2\,d\zeta,\\&\left|\int_0^t l(A,u,\rho)\,d\zeta\right|\leq Ch^{2s}\int_0^t\|u\|_{s+1}^2\,d\zeta+C\int_0^t|||\rho|||^2\,d\zeta,\\&\left|\int_0^t s(Q_hu,\rho)\,d\zeta\right|\leq Ch^{2s}\int_0^t\|u\|_{s+1}^2\,d\zeta+C\int_0^t|||\rho|||^2\,d\zeta.\end{aligned}$$

Combining these inequalities, we get

$$|||\rho|||^2 \le C\int_0^t |||\rho|||^2\, d\zeta + Ch^{2s}\|u\|_{s+1}^2 + Ch^{2s}\int_0^t \|u\|_{s+1}^2\, d\zeta.$$

Using the Gronwall inequality, we obtain the error estimate in H^1 norm. □

Next, we estimate $\|\rho\|$. Assume $\omega \in H^1(\Omega)$ is the exact solution of the elliptic problem with a Drichlet boundary condition

$$-\nabla\cdot(A\nabla\omega) = \rho_0 \quad \text{in } \Omega, \tag{5.3a}$$

$$\omega = 0 \quad \text{on } \partial\Omega. \tag{5.3b}$$

We presume that the dual problem has the $H^1(\Omega)$-regularity property

$$\|\omega\|_2 \le C\|\rho_0\|.$$

Theorem 5.2. *Let $u \in H^{s+1}(\Omega)$ be the exact solution of linear integro-differential equation without the term u_t. We have the following error estimate for ρ_0:*

$$\|\rho_0\| \le Ch^{s+1}\left(\|u\|_{s+1}^2 + \int_0^t \|u\|_{s+1}^2 d\zeta\right)^{\frac{1}{2}}, \qquad 0 \le t \le T. \tag{5.4}$$

Proof. Testing the elliptic problem (5.3a) against ρ_0, we have

$$\begin{aligned}\|\rho_0\|^2 &= (-\nabla\cdot(A\nabla\omega),\rho_0) = (A\nabla\omega,\nabla\rho_0) - \sum_{T\in\mathcal{T}_h}\langle A\nabla\omega\cdot n,\rho_0\rangle_{\partial T}\\ &= ((A-\overline{A})(\nabla\omega - R_h(\nabla\omega)),\nabla\rho_0) + (AR_h(\nabla\omega),\nabla\rho_0)\\ &\quad - \sum_{T\in\mathcal{T}_h}\langle A\nabla\omega\cdot n,\rho_0-\rho_b\rangle_{\partial T}\\ &= ((A-\overline{A})(\nabla\omega - R_h(\nabla\omega)),\nabla\rho_0) + a(A,Q_h\omega,\rho)\\ &\quad - s(Q_h\omega,\rho) - l(A,\omega,\rho),\end{aligned} \tag{5.5}$$

where using the fact $(\overline{A}(\nabla\omega - R_h\nabla\omega),\rho_0) = 0$ and $\sum_{T\in\mathcal{T}_h}\langle A\nabla\omega\cdot n,\rho_b\rangle = 0$. Considering Eq. (5.5) and taking $v = Q_h\omega$ in the error equation (5.2), we obtain

$$\begin{aligned}\|\rho_0\|^2 = &-\int_0^t (B\nabla_w\rho, R_h(\nabla\omega) - \nabla\omega)\, d\zeta - \int_0^t (\nabla_w\rho, B\nabla\omega)\, d\zeta\\ &-\int_0^t s(\rho,Q_h\omega)\, d\zeta + ((A-\overline{A})(\nabla\omega - R_h(\nabla\omega),\nabla\rho_0)\\ &+ l(A,\omega,\rho) - s(Q_h\omega,\rho) - ((A-\overline{A})(\nabla u - R_h(\nabla u)),\nabla(Q_0\omega))\\ &-\int_0^t ((B-\overline{B})(\nabla u - R_h(\nabla u)),\nabla(Q_0\omega))\, d\zeta + l(A,u,Q_h\omega)\\ &+\int_0^t l(B,u,Q_h\omega)\, d\zeta + s(Q_hu,Q_h\omega) + \int_0^t s(Q_hu,Q_h\omega)\, d\zeta.\end{aligned}$$

In order to get the estimates, we estimate the form term by term. Using the definition of the discrete weak gradient, the trace inequality and the property of projection yield

$$\begin{aligned}
&\left|\int_0^t (B\nabla_w\rho, R_h(\nabla\omega)-\nabla\omega)\, d\zeta\right| \le C\int_0^t \|\nabla_w\rho\|\|R_h(\nabla\omega)-\nabla\omega\|\, d\zeta \\
&\quad \le Ch\int_0^t \|\omega\|_2 |||\rho|||\, d\zeta, \\
&\left\|\int_0^t (\nabla_w\rho, B\nabla\omega)\, d\zeta\right\| = \left\|\int_0^t (\rho_0, \nabla\cdot(B\nabla\omega))\, d\zeta\right\| \le \int_0^t \|\rho_0\|\ \|\omega\|_2 d\zeta, \\
&|(A-\overline{A})(\nabla\omega - R_h(\nabla\omega)), \nabla\rho_0| \le Ch^2\|A\|_{1,\infty}\cdot\|\omega\|_2|||\rho|||,
\end{aligned}$$

and

$$\begin{aligned}
|s(Q_hu,Q_h\omega)| &\le C\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0u-u\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\omega-\omega+\omega-Q_b\omega\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\le Ch^{s+1}\|u\|_{s+1}\|\omega\|_2, \\
|l(A,u,Q_h\omega)| &= \Big|\sum_{T\in\mathcal{T}_h}\langle A(R_h(\nabla u)-\nabla u)\cdot\mathbf{n}, Q_0\omega-\omega+\omega-Q_b\omega\rangle_{\partial T}\Big| \\
&\le Ch^{s+1}\|u\|_{s+1}\|\omega\|_2.
\end{aligned}$$

Following the estimate of $|||\rho|||$, the definition of the discrete weak gradient, the trace inequality and the property of projection, we have

$$\begin{aligned}
&|s(Q_h\omega,\rho)| \le Ch\|\omega\|_2|||\rho|||, \qquad |l(A,\omega,\rho)| \le Ch\|\omega\|_2|||\rho|||, \\
&\left|\int_0^t ((B-\overline{B})(\nabla u - R_h(\nabla u)), \nabla(Q_0\omega))d\zeta\right| \le Ch\int_0^t \|\omega\|_2\|\rho\|\, d\zeta, \\
&|((A-\overline{A})(\nabla u - R_h(\nabla u)), \nabla(Q_0\omega))| \le Ch^{s+1}\|\omega\|_2\|u\|_{s+1},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^t s(Q_hu,Q_h\omega)d\zeta\right| \le Ch^{s+1}\int_0^t \|u\|_{s+1}\|\omega\|_2 d\zeta, \\
&\left|\int_0^t s(\rho,Q_h\omega)\, d\zeta\right| \le Ch\int_0^t \|\rho\|\|\omega\|_2 d\zeta, \\
&\left|\int_0^t l(B,u,Q_h\omega)d\zeta\right| \le Ch^{s+1}\int_0^t \|u\|_{s+1}\|\omega\|_2 d\zeta.
\end{aligned}$$

From the estimate of $|||\rho|||$, regularity property and the Cauchy-Schwarz inequality, we get

$$\|\rho_0\|^2 \le \frac{1}{2}\|\rho_0\|^2 + C\int_0^t \|\rho_0\|^2 d\zeta + Ch^{s+1}\left(\|u\|_{s+1}^2 + \int_0^t \|u\|_{s+1}^2 d\zeta\right).$$

According to the Gronwall inequality, we obtain

$$\|\rho_0\| \leq Ch^{s+1}\left(\|u\|_{s+1}^2+\int_0^t \|u\|_{s+1}^2 d\zeta\right)^{\frac{1}{2}}.$$

The proof of this theorem is completed. □

Similar to the proof of Theorem 5.1 and Theorem 5.2, we can obtain the estimate of ρ_t.

Theorem 5.3. *If $u_t \in L^\infty(0,T;H^{s+1}(\Omega))$, there is a positive C with $0\leq t\leq T$ such that*

$$|||\rho_t||| \leq Ch^s\left(\|u\|_{s+1}^2+\|u_t\|_{s+1}^2+\int_0^t (\|u\|_{s+1}^2+\|u_t\|_{s+1}^2)\, d\zeta\right)^{\frac{1}{2}},$$

$$\|(\rho_0)_t\| \leq Ch^{s+1}\left(\|u\|_{s+1}^2+\|u_t\|_{s+1}^2+\int_0^t (\|u\|_{s+1}^2+\|u_t\|_{s+1}^2)\, d\zeta\right)^{\frac{1}{2}}.$$

6 Numerical example

We present several numerical examples to verity the order of convergence with weak Galerkin finite element method by adding the stabilizer for the linear parabolic integro-differential equation (1.1a)-(1.1c).

Example 6.1. We solve (1.1a) over the square domain $\Omega=(0,1)\times(0,1)$ where

$$A=\begin{pmatrix}3+x-y & 1/2\\ 1/2 & 4-x+y\end{pmatrix},\qquad B=\begin{pmatrix}x+y & -1/2\\ -1/2 & x+y\end{pmatrix},$$
$$\Omega=(0,1)\times(0,1),$$

and the exact solution is chosen as

$$u(x,y,t)=e^{-t}x(1-x)y(1-y). \tag{6.1}$$

We use P_k weak Galerkin finite elements on rectangular grids where the first level grid is the domain itself and each grid is refined in to the half-sized grid to form the next level grid. We choose $\tau=10^{-4}$ and compute the solution up to $T=1$. The errors and the orders of convergence are listed in Table 1.

Example 6.2. We solve (1.1a) where

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad B=\beta_1\begin{pmatrix}0 & -1/2\\ -1/2 & 0\end{pmatrix},\qquad \beta_1=1 \quad \text{or} \quad 10,$$
$$f=1,\qquad g=0,\qquad \psi=0,\qquad \Omega\times(0,T)=(0,1)\times(0,1)\times(0,1).$$

We do not know the exact solution. The numerical solutions are plotted in Fig. 1, where we can see the effect of memory integral.

Table 1: Example 6.1. The errors and the orders of convergence for solving (6.1).

$\mathcal{T}_k$	$\|Q_hu-u_h\|_0$	h^n	$\|Q_hu-u_h\|_{1,h}$	h^n	$\|\|\|Q_hu-u_h\|\|\|$	h^n
	P_1 weak Galerkin finite element method					
4	0.1528E-02	1.90	0.7005E-02	1.08	0.4221E-02	1.34
5	0.3889E-03	1.97	0.3601E-02	0.96	0.1176E-02	1.84
6	0.9743E-04	2.00	0.1876E-02	0.94	0.3031E-03	1.96
	P_2 weak Galerkin finite element method					
4	0.6438E-04	3.07	0.1997E-02	2.04	0.1397E-02	2.27
5	0.7492E-05	3.10	0.4782E-03	2.06	0.2871E-03	2.28
6	0.9107E-06	3.04	0.1161E-03	2.04	0.5943E-04	2.27
	P_3 weak Galerkin finite element method					
2	0.1048E-02	3.70	0.1140E-01	2.55	0.1315E-01	2.93
3	0.6760E-04	3.95	0.1529E-02	2.90	0.1657E-02	2.99
4	0.4138E-05	4.03	0.1891E-03	3.02	0.1967E-03	3.07
	P_4 weak Galerkin finite element method					
2	0.9909E-04	5.06	0.1826E-02	4.07	0.2976E-03	2.94
3	0.3014E-05	5.04	0.1079E-03	4.08	0.2582E-04	3.53
4	0.1551E-06	4.28	0.6457E-05	4.06	0.1910E-05	3.76

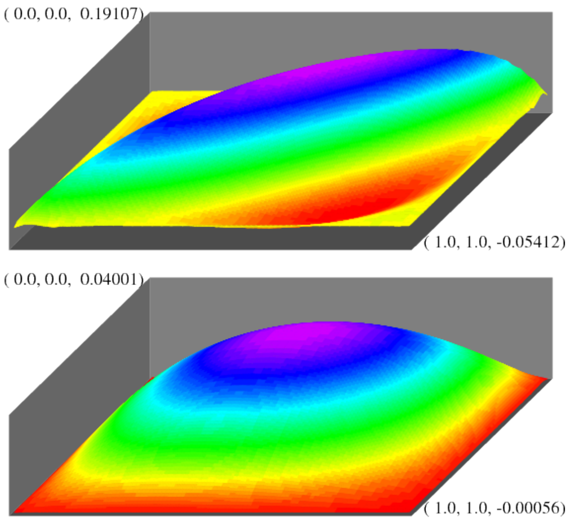

Figure 1: Example 6.2. The weak Galerkin solution for $\beta=1$ (bottom) and for $\beta=10$ (top).

7 Conclusions

In this paper, we developed another weak Galerkin finite element method with a stabilizer, which provides more options to choose elements of partition, for solving linear parabolic and primary integro-differential problems. The semi-discrete and fully-discrete weak Galerkin finite element schemes were constructed. The semi-discrete WG scheme was proved to be stable with respect to the right hand side. The fully-discrete WG scheme was discretized by the backward Euler method. The optimal orders of convergence were obtained in L^2 and H^1 norms. Numerical examples confirmed theoretical analysis.

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