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Superconvergence Analysis for the Maxwell's Equations in Debye Medium with a Thermal Effect

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Abstract. In this paper, a mixed finite element method is investigated for the Maxwell's equations in Debye medium with a thermal effect. In particular, in two dimensional case, the zero order Nédélec element $(Q_{01}\times Q_{10})$, the piecewise constant space Q_0 element, and the bilinear element Q_{11} are used to approximate the electric field **E** and the polarization electric field **P**, the magnetic field H, and the temperature field u, respectively. With the help of the high accuracy results, mean-value technique and interpolation postprocessing approach, the convergent rate $\mathcal{O}(\tau+h^2)$ for global superconvergence results are obtained under the time step constraint $\tau=\mathcal{O}(h^{1+\gamma})$, $\gamma>0$ by using the linearized backward *Euler* finite element discrete scheme. At last, a numerical experiment is given to verify the theoretical analysis and the validity of our method.

AMS subject classifications: 65N30, 65N15

Key words: Maxwell's equations, thermal effect, error analysis, superconvergence.

1 Introduction

In this paper, we consider the following Maxwell's equations:

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\qquad \nabla\times\mathbf{H}=-\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J},$$
$$\nabla\cdot\mathbf{D}=\rho,\qquad \nabla\cdot\mathbf{B}=0,$$

where **E** and **H** denote the strengths of the electric and magnetic fields, respectively. **D** and **B** are the electric and magnetic flux densities, respectively. **J** and ρ represent the

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current density and the density of free electric charge, respectively. The above equations will be supplemented with the constitutive laws by:

$$\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}, \qquad \mathbf{B}=\mu_0\mathbf{H}+\mu_0\mathbf{M},$$

where **P** and **M** represent the electric and magnetic polarization, respectively. ϵ_0 and μ_0 are the electric permittivity of free space and the magnetic permeability, respectively. We assume $\mathbf{M}=0$ since we can choose to ignore the magnetic effect among the dielectric materials.

Debye medium is one of basic physical concepts when one investigates dielectric in electromagnetic theory and materials science [22]. It is a kind of isotropic dispersive medium, and its permittivity and conductivity are functions of frequency. With the help of the polarization and dielectric relaxation, one can establish phenomenological theory in Debye medium. That is to say, in the process of polarization, microscopic particles complex energy exchange actions can be taken into the following dielectric time parameters. Therefore, numerical studies of Maxwell's equations in Debye medium have attracted considerable attention.

The linear polarization representation originates from the model proposed by Debye [3]. Similar to this representation, in this paper, we consider a linear polarization model

$$\mathbf{P}_t+\frac{1}{t_0}\mathbf{P}=\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}\mathbf{E},$$

where ϵ_s, ϵ_∞ and t_0 stand by the static relative permittivity, the value of permittivity for an extremely high frequency field and the relaxation time of the dielectric materials, respectively.

Considering the effect of temperature field on electromagnetic field, we use Ohm's law $\mathbf{J}=\sigma(u)\mathbf{E}$ to describe the system, The unknown u is the temperature, the local density of Joule's heat generated by intensive electric waves equals [21]

$$\mathbf{E}\cdot\mathbf{J}=\mathbf{E}\cdot\sigma(u)\mathbf{E}=\sigma(u)|\mathbf{E}|^2.$$

Thus, from Fourier's law and the conservation of energy, we see that u satisfies

$$u_t-\nabla\cdot(k\nabla u)=\sigma(u)|\mathbf{E}|^2r,$$

where k is the coefficient of thermal conductivity and other physical constants such as density and specific heat have been normalized.

Throughout the paper, we suppose that σ is Lipschitz continuous with respect to u, which satisfies

$$0<\sigma_{\min}\le\sigma\le\sigma_{\max}.$$

Assumed that $\Omega\subset R^2$ is a bounded domain with Lipschitz boundary, we consider the following coupling model

$$\epsilon_0\mathbf{E}_t+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}\mathbf{E}+\sigma(u)\mathbf{E}-\nabla\times H-\frac{1}{t_0}\mathbf{P}=0,\qquad (X,t)\in\Omega\times(0,T], \tag{1.1a}$$

$$\mu H_t+\nabla\times\mathbf{E}=0,\qquad (X,t)\in\Omega\times(0,T], \tag{1.1b}$$

$$\mathbf{P}_t+\frac{1}{t_0}\mathbf{P}=\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}\mathbf{E},\qquad (X,t)\in\Omega\times(0,T], \tag{1.1c}$$

$$u_t-\nabla\cdot(k\nabla u)=\sigma(u)|\mathbf{E}|^2,\qquad (X,t)\in\Omega\times(0,T], \tag{1.1d}$$

with the initial date

$$\mathbf{E}(X,0)=\mathbf{E}_0,\quad H(X,0)=H_0,\quad \mathbf{P}(X,0)=\mathbf{P}_0,\quad u(X,0)=u_0,\quad X\in\Omega, \tag{1.2}$$

and the perfectly electric boundary condition

$$\mathbf{E}\times\mathbf{n}=0,\qquad u(X,t)=0,\qquad (X,t)\in\partial\Omega\times[0,T], \tag{1.3}$$

where $\mathbf{n}$ is the outward normal vector on $\partial\Omega$.

As we know, superconvergence analysis has been an interesting field. A sizable amount of researches have been done on the superconvergence of finite element methods for many types of PDEs, such as, the second order elliptic equation [18], parabolic equation [36], Stokes equations [20], nonlinear Sobolev equation [29, 30], Schrödinger equation [5], parabolic equationswith integral two-space-variables condition [6] and so on. There have been a few theoretical results on superconvergence analysis for Maxwell's equation. In [24], the authors studied the superconvergence of the Maxwell's equations in 1994 for the first time. Later, in [27], the authors used the integral identity technique to study this problem once more and improved the result in [24]. The similar result was provied for 2-D and 3-D Maxwell's equations in [7, 8, 11–13, 16, 17]. In [4], the authors discussed the superconvergence of second and third order rectangular edge elements. In [31], the authors studied the superconvergence of nonconforming mixed finite element methods for 3-D time-dependent Maxwell's equations in isotropic cold plasma media. In recent years, Maxwell's equations in Debye medium [26–28,32] have been studied the convergence and superconvergence properties of the nonconforming finite element. Now stochastic collocation methods for Maxwell's equations with random inputs [10] are becoming another popular issue.

The relevant models of Maxwell's equations in the above mentioned studies are linear, and the influence of temperature field on electromagnetic field has been ignored in the process of research. In [35], the coupling system of temperature field and electromagnetic field is studied for the first time, and it is proved that when $\sigma(u)$ is nonnegative and bounded, the weak solution $\mathbf{E}$, $\mathbf{H}$, u is existence and uniqueness when $\mathbf{E}(X,t)$, $\mathbf{H}(X,t)\in L^\infty(0,T;(L^2(\Omega))^3)$, $u(X,t)\in L^q(0,T;W_0^{1,q}(\Omega))$, $q\in(1,5/4)$. However, no relevant reports has been found on the finite element method. The similar analysis of existence and uniqueness for nonlinear Maxwell's equations can be found in [33,34].

In this paper, we will focus on the superclose and superconvergence properties of the nonlinear coupled model (1.1a)-(1.3). For convenience, this paper only discusses the case in 2-D. The article is organized as follows. In Section 2, we introduce some notations, the mixed finite element scheme and the variational problem. In Section 3, we give the linearized backward Euler fully discrete scheme and deduce the superclose estimates of order $\mathcal{O}(\tau+h^2)$, where $\tau=\mathcal{O}(h^{1+\gamma})$, $\gamma>0$. In Section 4, the $\mathcal{O}(\tau+h^2)$ order global superconvergence results are obtained with the help of the interpolation postprocessing technique of the coupled model. At last, some numerical results are provided to verify the theoretical analysis, and show the efficiency of the method. Furthermore, it can be verified that when $\gamma>1/2$, the conclusion in this paper can be extended to the 3-D after replacing the mixed Nédélec elements.

2 Construction of mixed finite elements and variational formulation

We need the following Sobolev spaces

$$\begin{aligned}
&H(curl;\Omega):=\{\mathbf{w}=(w_1,w_2)\in(L^2(\Omega))^2:\nabla\times\mathbf{w}\in(L^2(\Omega))^2\},\\
&H_0(curl;\Omega):=\{\mathbf{w}\in H(curl;\Omega):\mathbf{n}\times\mathbf{w}=0 \text{ on } \partial\Omega\},\\
&H^s(curl;\Omega):=\{\mathbf{w}\in(H^s(\Omega))^2:\nabla\times\mathbf{w}\in(H^s(\Omega))^2\},
\end{aligned}$$

where $s>0$,and $\mathbf{n}$ is the unit outer normal to Ω.

The above spaces are equipped with the norms, respectively,

$$\begin{aligned}
&\|\mathbf{w}\|^2_{H(curl;\Omega)}:=\|\mathbf{w}\|^2_{(L^2(\Omega))^2}+\|\nabla\times\mathbf{w}\|^2_{(L^2(\Omega))^2},\\
&\|\mathbf{w}\|^2_{H^s(curl;\Omega)}:=\|\mathbf{w}\|^2_{H^s(curl;\Omega)}+\|\nabla\times\mathbf{w}\|^2_{H^s(curl;\Omega)}.
\end{aligned}$$

Now, we consider that the weak formulation of system (1.1a)-(1.3) in two-dimention. For any $t>0$, find $(\mathbf{E}(t),\ H(t),\ \mathbf{P}(t),\ u(t))\in H_0(curl;\Omega)\times L^2(\Omega)\times H(curl;\Omega)\times H^1_0(\Omega)$, such that

$$\begin{aligned}
&\epsilon_0(\mathbf{E}_t,\phi)+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\mathbf{E},\phi)+(\sigma(u)\mathbf{E},\phi)-(H,\nabla\times\phi)\\
&\qquad-\frac{1}{t_0}(\mathbf{P},\phi)=0, && \forall\phi\in H_0(curl;\Omega), \quad (2.1a)\\
&\mu(H_t,\psi)+(\nabla\times\mathbf{E},\psi)=0, && \forall\psi\in L^2(\Omega), \quad (2.1b)\\
&(\mathbf{P}_t,\mathbf{w})+\frac{1}{t_0}(\mathbf{P},\mathbf{w})=\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\mathbf{E},\mathbf{w}), && \forall\mathbf{w}\in H(curl;\Omega), \quad (2.1c)\\
&(u_t,v)+(k\nabla u,\nabla v)=(\sigma(u)|\mathbf{E}|^2,v), && \forall v\in H^1_0(\Omega). \quad (2.1d)\\
&\mathbf{E}(X,0)=\mathbf{E}_0,\quad \mathbf{H}(X,0)=\mathbf{H}_0,\quad \mathbf{P}(X,0)=\mathbf{P}_0,\quad u(X,0)=u_0,\quad X\in\Omega. && (2.1e)
\end{aligned}$$

Since **P** in (2.1a)-(2.1d) is the ordinary differential equation with respect to **E**, the existence and uniqueness is similar to [35], and we do not pay much attention to it here.

Assume that Ω is a rectangle in the $X=(x,y)$ plane with edges parallel to the coordinate axes. Let $\mathcal{T}_h$ be a rectangular subdivision of Ω satisfying the regular condition. Given $K\in\mathcal{T}_h$, we denote the lengths of edges parallel to x -axis and y -axis by $2h_{x,K}$, $2h_{y,K}$, respectively. Set

$$h_K=\max\{h_{x,K},h_{y,K}\} \quad \text{and} \quad h=\max_{K\in\mathcal{T}_h}\{h_K\}.$$

The finite element spaces $\mathbf{N}_h$, W_h, V_h are defined by

$$\begin{aligned}
&\mathbf{N}_h=\{\mathbf{E}=(E_1,E_2)\in H(curl;\Omega);\ \mathbf{E}|_K\in Q_{01}(K)\times Q_{10}(K),\ \forall K\in\mathcal{T}_h\},\\
&W_h=\{w\in L^2(\Omega);\ w|_K\in Q_0(K),\ \forall K\in\mathcal{T}_h\},\\
&V_h=\{v;v|_K\in Q_{11}(K),\ \forall K\in\mathcal{T}_h\}, \qquad V_0^h=\{v;v\in V_h,\ v|_{\partial\Omega}=0\},
\end{aligned}$$

where

$$\begin{aligned}
&H(curl;\Omega)=\left\{\mathbf{E}\in[L^2(\Omega)]^2,\ \nabla\times\mathbf{E}=\frac{\partial E_2}{\partial x}-\frac{\partial E_1}{\partial y}\right\},\\
&Q_{ij}=span\{x^ry^s,\ 0\le r\le i,\ 0\le s\le j\}.
\end{aligned}$$

For $v\in H^2(\Omega)$, $\mathbf{E}=(E^1,E^2)\in(H^1(\Omega))^2$, we define the associated interpolation operator $\mathbf{\Pi}_h$, R_h, I_h as

$$\begin{aligned}
&\mathbf{\Pi}_h:\mathbf{E}\in(H^1(\Omega))^2\to\mathbf{\Pi}_h\mathbf{E}\in\mathbf{N}_h, \quad \mathbf{\Pi}_h|_K=\mathbf{\Pi}_K, \quad \int_{l_i}(\mathbf{E}-\mathbf{\Pi}_K\mathbf{E})\cdot\mathbf{n}_i ds=0,\\
&R_h:w\in L^2(\Omega)\to R_hw\in W_h, \quad R_h|_K=R_K, \quad \int_K(w-R_Kw)rds=0, \quad \forall r\in W_h,\\
&I_h:v\in H^2(\Omega)\to I_hv\in V_h, \quad I_h|_K=I_K, \quad I_Kv(a_i)=v(a_i), \qquad i=1,2,3,4,
\end{aligned}$$

respectively, where $\mathbf{n}_i$ is the unit tangent vector of l_i.

We can cite the following interpolation error estimate in [1,19,23]

Lemma 2.1. *1. For any $\mathbf{E}\in H^\alpha(curl;\Omega)$, $\frac{1}{2}<\alpha\le 1$, we have*

$$\|\mathbf{E}-\mathbf{\Pi}_h\mathbf{E}\|_0+\|\nabla\times(\mathbf{E}-\mathbf{\Pi}_h\mathbf{E})\|_0\le Ch^\alpha\|\mathbf{E}\|_{H^\alpha(curl;\Omega)},$$

2. For $0\le m\le l-1$, if m and p satisfy $m\ge 2$, $p\le 1$ or $m=1$, $p>6/5$, for $\forall\mathbf{E}\in(W^{m+1,p}(K))^2$, we have

$$\|\mathbf{E}-\mathbf{\Pi}_h\mathbf{E}\|_{L^p(K)}\le Ch^{m+1}\|D^{m+1}\mathbf{E}\|_{L^p(K)}.$$

In addition, if $m=0$, $p\le 2$, for $\forall\mathbf{E}\in(W^{1,p}(K))^2$, $D\mathbf{curl}\mathbf{E}\in L^s(K)$, we have

$$\|\mathbf{E}-\mathbf{\Pi}_h\mathbf{E}\|_{L^p(K)}\le Ch(\|D\mathbf{E}\|_{L^p(K)}+h|K|^{\frac{1}{p}-\frac{1}{s}}\|D\mathbf{curl}\mathbf{E}\|_{L^s(K)}).$$

We also have the following the interpolation error estimate: $v\in H^{m+1}(\Omega)$ *for* $1\le m\le k$, *we have*

$$\|v-I_hv\|_{L^2(\Omega)}+h\|\nabla(v-I_hv)\|_{L^2(\Omega)}\le Ch^{m+1}\|v\|_{H^{m+1}(\Omega)},$$

where C *is a positive constant independent of both the mesh size* h *and the time step* τ.

Define $\mathbf{ff}=(\alpha_1,\alpha_2)\triangleq\mathbf{E}-\mathbf{\Pi}_h\mathbf{E}=(E_1-\mathbf{\Pi}_hE_1,E_2-\mathbf{\Pi}_hE_2)$, then we recall the following two lemmas which can be found in [14] and plays an important role later.

Lemma 2.2. *Assume that* $u\in H^3(\Omega)$, *there hold*

$$\int_K\nabla(u-I_hu)\nabla vdxdy=\mathcal{O}(h^2)|u|_{3,K}\|v\|_{1,K},\qquad\forall v\in V_h.$$

In addition, if $u\in H^4(\Omega)$, *we have*

$$\int_K\nabla(u-I_hu)\nabla vdxdy=\mathcal{O}(h^2)|u|_{4,K}\|v\|_{0,K},\qquad\forall v\in V_h^0.\tag{2.2}$$

Lemma 2.3. *For* $\mathbf{E}\in(H^2(\Omega))^2$, $w\in L^2(\Omega)$, *we have*

$$\begin{aligned}
&(\mathbf{E}-\mathbf{\Pi}_h\mathbf{E},\mathbf{p})=\mathcal{O}(h^2)\|\mathbf{E}\|_2\|\mathbf{p}\|_0, &&\forall\mathbf{p}\in\mathbf{N}_h,\\
&(\nabla\times(\mathbf{E}-\mathbf{\Pi}_h\mathbf{E}),r)=0, &&\forall r\in W_h,\\
&(w-R_hw,\nabla\times\mathbf{p})=0, &&\forall\mathbf{p}\in\mathbf{N}_h,\\
&(w-R_hw,r)=0, &&\forall r\in W_h.
\end{aligned}$$

Lemma 2.4. *For all* $\mathbf{E}=(E_1,E_2)\in(H^2(\Omega))^2$, *there hold*

$$(\alpha_j,v)=\mathcal{O}(h^2)\|E_j\|_2\|v\|_1,\qquad(j=1,2),\qquad\forall v\in V_h.$$

Proof. First of all, we introduce two error functions:

$$E(x)=\frac{1}{2}((x-x_K)^2-h_{x,K}^2),\qquad F(y)=\frac{1}{2}((y-y_K)^2-h_{y,K}^2).$$

$\forall v\in V_h$, it easy to see that

$$v(x,y)=v(x_K,y_K)+(x-x_K)v_x(x_K,y_K)+(y-y_K)v_y(x_K,y_K)+(x-x_K)(y-y_K)v_{xy}$$

holds.

By using the properties of $E(x)$, $F(y)$ and the definition of the element $Q_{01}\times Q_{10}$, we have

$$\int_K\alpha_1dxdy=\int_K\alpha_1F''(y)dxdy=-\int_K\alpha_{1y}F'(y)dxdy=\int_KE_{1yy}F(y)dxdy,$$

which causes

$$\begin{aligned}\int_K\alpha_1v(x_K,y_K)dxdy=\int_KE_{1yy}F(y)[&v(x,y)-(x-x_K)v_x(x_K,y_K)\\&-(y-y_K)v_y(x_K,y_K)-(x-x_K)(y-y_K)v_{xy}]dxdy.\end{aligned}$$

With the help of interpolation error estimation and inverse inequality [9,14], we have

$$\int_K \alpha_1 v(x_K,y_K)dxdy = \mathcal{O}(h^2)|E_1|_2\|v\|_1. \tag{2.3}$$

Similarly, we have

$$\sum_K \int_K \alpha_1 (x-x_K) v_x(x_K,y_K)dxdy = \mathcal{O}(h^2)|E_1|_2\|v\|_1, \tag{2.4a}$$

$$\sum_K \int_K \alpha_1 (y-y_K) v_y(x_K,y_K)dxdy = \mathcal{O}(h^2)|E_1|_2\|v\|_1, \tag{2.4b}$$

$$\sum_K \int_K \alpha_1 (x-x_K)(y-y_K) v_{xy}dxdy = \mathcal{O}(h^2)|E_1|_2\|v\|_1. \tag{2.4c}$$

By (2.3)-(2.4), we obtain

$$(\alpha_1,v) = \mathcal{O}(h^2)\|E_1\|_2\|v\|_1.$$

Similarly,

$$(\alpha_2,v) = \mathcal{O}(h^2)\|E_2\|_2\|v\|_1.$$

The proof is completed. □

3 Backward Euler fully discrete scheme

In this section, we can turn our attention to the discrete scheme. For positive integer N, let $0=t_0<t_1<\cdots<t_N=T$ be a partition of the time interval $[0,T]$ with step length $\tau=\frac{T}{N}$ for some positive integer N. For a sequence of function $\{\phi^n\}_{n=0}^N$, we denote $\partial_\tau \phi^n = \frac{\phi^n-\phi^{n-1}}{\tau}$. In the same manner of [28], we use the following linearized backward Euler discrete scheme: for $n=1,2,\cdots,N-1$, seek $(\mathbf{E}_h^n,H_h^n,\mathbf{P}_h^n,u_h^n):\in \mathbf{N}_h\times W_h\times \mathbf{N}_h\times V_h$, such that

$$\begin{aligned}
&\epsilon_0(\partial_\tau \mathbf{E}_h^n,\phi_h)+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\mathbf{E}_h^n,\phi_h)+(\sigma(u_h^{n-1})\mathbf{E}_h^n,\phi_h)\\
&\quad -(H_h^n,\nabla\times\phi_h)-\frac{1}{t_0}(\mathbf{P}_h^n,\phi_h)=0, && \forall \phi_h\in\mathbf{N}_h, && \text{(3.1a)}\\
&\mu(\partial_\tau H_h^n,\psi_h)+(\nabla\times\mathbf{E}_h^n,\psi_h)=0, && \forall\psi_h\in W_h, && \text{(3.1b)}\\
&(\partial_\tau \mathbf{P}_h^n,\mathbf{w}_h)+\frac{1}{t_0}(\mathbf{P}_h^n,\mathbf{w}_h)=\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\mathbf{E}_h^{n-1},\mathbf{w}_h), && \forall\mathbf{w}_h\in\mathbf{N}_h, && \text{(3.1c)}\\
&(\partial_\tau u_h^n,v_h)+(k\nabla u_h^n,\nabla v_h)=(\sigma(u_h^{n-1})|\mathbf{E}_h^n|^2,v_h), && \forall v_h\in V_h, && \text{(3.1d)}
\end{aligned}$$

with the initial approximations:

$$\mathbf{E}_h^0=\mathbf{\Pi}_h\mathbf{E}_0,\qquad H_h^0=R_hH_0,\qquad \mathbf{P}_h^0=\mathbf{\Pi}_h\mathbf{P}_0,\qquad u_h^0=\mathbf{I}_hu_0.$$

We give the existence and uniqueness of the solutions of Eqs. (3.1a)-(3.1d).

Theorem 3.1. *For any $n=1,2,\cdots,M$, there exists a unique solution $(\mathbf{E}_h^n,H_h^n,\mathbf{P}_h^n,u_h^n):\in \mathbf{N}_h\times W_h\times \mathbf{N}_h\times V_h$, to solve Eqs.* (3.1a)-(3.1d).

Proof. Notice that $\mathbf{E}_h^n$ and $\mathbf{P}_h^n$ are chosen from the same finite element space, here $\mathbf{P}_h^n$ can be computed as follows:

$$\mathbf{P}_h^n=\Big[\frac{1}{\tau}\mathbf{P}_h^{n-1}+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}\mathbf{E}_h^{n-1}\Big]\Big(\frac{1}{\tau}+\frac{1}{t_0}\Big)^{-1}. \tag{3.2}$$

Substituting (3.2) into (3.1a), we can rewrite this equation as follows:

$$\begin{aligned}&\Big(\frac{\epsilon_0}{\tau}+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}\Big)(\mathbf{E}_h^n,\phi_h)+(\sigma(u_h^{n-1})\mathbf{E}_h^n,\phi_h)-(H_h^n,\nabla\times\phi_h)\\ =&\frac{1}{t_0+\tau}(\mathbf{P}_h^{n-1},\phi_h)+\Big(\frac{\epsilon_0}{\tau}+\frac{\tau\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0(t_0+\tau)}\Big)(\mathbf{E}_h^{n-1},\phi_h),\qquad \forall\phi_h\in\mathbf{N_h}.\end{aligned} \tag{3.3}$$

On the other hand, we can rewrite (3.1b) as follows

$$\frac{\mu}{\tau}(H_h^n,\psi_h)+(\nabla\times\mathbf{E}_h^n,\psi_h)=\frac{\mu}{\tau}(H_h^{n-1},\psi_h),\qquad \forall\psi_h\in W_h. \tag{3.4}$$

Hence, the backward Euler mixed finite element scheme for (3.1a)-(3.1c) can be realized in practice as follows: at each time step, we first solve a system of (3.3)-(3.4) for $\mathbf{E}_h^n$ and H_h^n, then update $\mathbf{P}_h^n$ by (3.2).

Finally, notice that the coefficient matrix for the system of (3.3)-(3.4) can be written as

$$Q\equiv\begin{pmatrix} A & -B\\ B' & D\end{pmatrix}, \tag{3.5}$$

where the matrices

$$A=\Big(\frac{\epsilon_0}{\tau}+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}\Big)(\mathbf{N_h},\mathbf{N_h})+(\sigma^{\frac12}(u_h^{n-1})\mathbf{N_h},\sigma^{\frac12}(u_h^{n-1})\mathbf{N_h}), \tag{3.6a}$$

$$B=(W_h,\nabla\times\mathbf{N_h}),\qquad D=\frac{\mu}{\tau}(W_h,W_h), \tag{3.6b}$$

and B' denotes the transpose of matrix B. Here, the determinant of Q can be obtained as

$$det(Q)=det(A)det(D+B'A^{-1}B),$$

which is guaranteed to be non-zero. Hence the system of (3.3)-(3.4) is guaranteed to have a unique solution $(\mathbf{E}_h^n,H_h^n)$ at each time step.

The equation of (3.1d) can be written as:

$$(U_h^n,v_h)+\tau(k\nabla U_h^n,\nabla v_h)=\tau(\sigma(U_h^{n-1})|\mathbf{E}_h^n|^2,v_h)+(U_h^{n-1},v_h),\qquad \forall v_h\in V_h, \tag{3.7}$$

it is easy to see that (3.7) have a unique solution by Lax-Milgram Lemma. The proof is completed. □

Theorem 3.2. *Let* $(\mathbf{E}^n, H^n, \mathbf{P}^n, u^n)$ *and* $(\mathbf{E}_h^n, \mathbf{P}_h^n, H_h^n, u_h^n)$ *be the solutions of the problem* (2.1a)-(2.1d) *and* (3.1a)-(3.1d) *at time* $t=t_n$*, respectively. Assume that*

$$\begin{aligned}
&\mathbf{E}\in L^\infty(0,T;H^2(curl;\Omega))\cap(W^{1,\infty}(\Omega))^2, \qquad \mathbf{P}\in L^\infty(0,T;H^2(curl;\Omega)),\\
&H\in L^\infty(0,T;L^2\Omega), \qquad u\in L^2(0,T;H^4(\Omega)),\\
&\mathbf{E}_t,\ \mathbf{P}_t\in L^\infty(0,T;H^2(curl;\Omega)), \qquad u_t\in L^\infty(0,T;H^2(\Omega)),\\
&\mathbf{E}_{tt},\ \mathbf{P}_{tt}\in L^\infty(0,T;(L^2(\Omega))^2), \qquad H_{tt}\in L^\infty(0,T;L^2(\Omega)), \qquad u_{tt}\in L^\infty(0,T;H_0^1(\Omega)).
\end{aligned}$$

We have the following superclose estimates

$$\begin{aligned}
&\max_{1\le n\le N}(\|\mathbf{E}_h^n-\mathbf{\Pi}_h\mathbf{E}^n\|_0+\|H_h^n-\mathbf{R}_hH^n\|_0+\|\mathbf{P}_h^n\\
&\qquad -\mathbf{\Pi}_h\mathbf{P}^n\|_0+\|\nabla(u_h^n-\mathbf{I}_hu^n)\|_0\le C(\tau+h^2).
\end{aligned} \tag{3.8}$$

Proof. Let

$$\begin{aligned}
&\mathbf{E}^n-\mathbf{E}_h^n=\mathbf{E}^n-\mathbf{\Pi}_h\mathbf{E}^n+\mathbf{\Pi}_h\mathbf{E}^n-\mathbf{E}_h^n=:\alpha^n+\beta^n,\\
&H^n-H_h^n=H^n-\mathbf{R}_hH^n+\mathbf{R}_hH^n-H_h^n=:\xi^n+\eta^n,\\
&\mathbf{P}^n-\mathbf{P}_h^n=\mathbf{P}^n-\mathbf{\Pi}_h\mathbf{P}^n+\mathbf{\Pi}_h\mathbf{P}^n-\mathbf{P}_h^n=:\theta^n+\gamma^n,\\
&u^n-u_h^n=u^n-I_hu^n+I_hu^n-u_h^n=:\lambda^n+\zeta^n.
\end{aligned}$$

At every time level n, we can get the following error equations

$$\begin{aligned}
&\epsilon_0\Big(\frac{\beta^n-\beta^{n-1}}{\tau},\phi_h\Big)+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\beta^n,\phi_h)+(\sigma(u^{n-1})\beta^n,\phi_h)-(\eta^n,\nabla\times\phi_h)-\frac{1}{t_0}(\gamma^n,\phi_h)\\
&=-\epsilon_0\Big(\frac{\alpha^n-\alpha^{n-1}}{\tau},\phi_h\Big)+\epsilon_0(R_1^n,\phi_h)-\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\alpha^n,\phi_h)\\
&\quad+(\xi^n,\nabla\times\phi_h)+\frac{1}{t_0}(\theta^n,\phi_h)-(\sigma(u^{n-1})\alpha^n,\phi_h)-((\sigma(u^{n-1})-\sigma(u_h^{n-1}))\mathbf{E}_h^n,\phi_h)\\
&\quad-((\sigma(u^n)-\sigma(u^{n-1}))\mathbf{E}^n,\phi_h),
\end{aligned} \tag{3.9a}$$

$$\mu\Big(\frac{\eta^n-\eta^{n-1}}{\tau},\psi_h\Big)+(\nabla\times\beta^n,\psi_h)=-\mu\Big(\frac{\xi^n-\xi^{n-1}}{\tau},\psi_h\Big)+\mu(R_2^n,\psi_h)-(\nabla\times\alpha^n,\psi_h), \tag{3.9b}$$

$$\begin{aligned}
&\Big(\frac{\gamma^n-\gamma^{n-1}}{\tau},w_h\Big)+\frac{1}{t_0}(\gamma^n,w_h)\\
&=-\Big(\frac{\theta^n-\theta^{n-1}}{\tau},w_h\Big)+(R_3^n,w_h)-\frac{1}{t_0}(\theta^k,w_h)+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\mathbf{E}^n-\mathbf{E}^{n-1},w_h)\\
&\quad+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\alpha^{n-1},w_h)+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\beta^{n-1},w_h),
\end{aligned} \tag{3.9c}$$

$$\begin{aligned}
&\Big(\frac{\zeta^n-\zeta^{n-1}}{\tau},v_h\Big)+(k\nabla\zeta^n,\nabla v_h)\\
&=-\Big(\frac{\lambda^n-\lambda^{n-1}}{\tau},v_h\Big)-(k\nabla\lambda^n,\nabla v_h)+((\sigma(u^{n-1})-\sigma(u_h^{n-1}))|\mathbf{E}_h^n|^2,v_h)\\
&\quad+(\sigma(u^{n-1})(\mathbf{E}^n+\mathbf{E}_h^n)(\mathbf{E}^n-\mathbf{E}_h^n),v_h)+((\sigma(u^n)-\sigma(u^{n-1}))|\mathbf{E}^n|^2,v_h)+(R_4^n,v_h),
\end{aligned} \tag{3.9d}$$

where, $R_1^n = D_\tau \mathbf{E}^n - \mathbf{E}_t^n$, $R_2^n = D_\tau H^n - H_t^n$, $R_3^n = D_\tau \mathbf{P}^n - \mathbf{P}_t^n$, $R_4^n = D_\tau u^n - u_t^n$, and

$$\|R_1^n\|_0^2 = \|\tau^{-1}\int_{t_{n-1}}^{t_n}(t_{n-1}-t)\mathbf{E}_{tt}(t)dt\|_0^2 \le C\tau\int_{t_{n-1}}^{t_n}\|\mathbf{E}_{tt}(t)\|_0^2 dt.$$

Similarly,

$$\|R_2^n\|_0^2 \le C\tau\int_{t_{n-1}}^{t_n}\|H_{tt}(t)\|_0^2 dt,$$
$$\|R_3^n\|_0^2 \le C\tau\int_{t_{n-1}}^{t_n}\|\mathbf{P}_{tt}(t)\|_0^2 dt,$$
$$\|R_4^n\|_0^2 \le C\tau\int_{t_{n-1}}^{t_n}\|u_{tt}(t)\|_0^2 dt.$$

Choosing $\phi_h = \beta^n$ in Eq. (3.9a) and $\psi_h = \eta^n$ in Eq. (3.9b), adding the results together, we have

$$\begin{aligned}
&\epsilon_0\Big(\frac{\beta^n-\beta^{n-1}}{\tau},\beta^n\Big)+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\beta^n,\beta^n)+(\sigma(u^{n-1})\beta^n,\beta^n)+\mu\Big(\frac{\eta^n-\eta^{n-1}}{\tau},\eta^n\Big)\\
=&-\epsilon_0\Big(\frac{\alpha^n-\alpha^{n-1}}{\tau},\beta^n\Big)+\epsilon_0(R_1^n,\beta^n)-\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\alpha^n,\beta^n)\\
&+(\xi_n,\nabla\times\beta^n)+\frac{1}{t_0}(\theta^n,\beta^n)+\frac{1}{t_0}(\gamma^n,\beta^n)-\mu\Big(\frac{\xi^n-\xi^{n-1}}{\tau},\eta^n\Big)\\
&+\mu(R_2^n,\eta_n)-(\nabla\times\alpha^n,\eta^n)-(\sigma(u^{n-1})\alpha^n,\beta^n)-((\sigma(u^{n-1})\\
&-\sigma(u_h^{n-1}))\mathbf{E}_h^n,\beta^n)-((\sigma(u^n)-\sigma(u^{n-1}))\mathbf{E}^n,\beta^n)\triangleq\sum_{i=1}^{12}A_i.
\end{aligned}\tag{3.10}$$

Now, we deal with the left term of (3.10),

$$\begin{aligned}
&\epsilon_0\Big(\frac{\beta^n-\beta^{n-1}}{\tau},\beta^n\Big)+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\beta^n,\beta^n)+(\sigma(u^{n-1})\beta^n,\beta^n)+\mu\Big(\frac{\eta^n-\eta^{n-1}}{\tau},\eta^n\Big)\\
\ge&\frac{\epsilon_0}{2\tau}(\|\beta^n\|_0^2-\|\beta^{n-1}\|_0^2)+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}\|\beta^n\|_0^2+\sigma_{\min}\|\beta^n\|_0^2+\frac{\mu}{2\tau}(\|\eta^n\|_0^2-\|\eta^{n-1}\|_0^2).
\end{aligned}$$

For the right term of (3.10), using Schwarz inequality, ϵ-Young inequality and Lemma 2.3, we have the following error estimates

$$\begin{aligned}
A_1\le&(Ch^2\tau^{-1}\|\mathbf{E}^n\|_2-Ch^2\tau^{-1}\|\mathbf{E}^{n-1}\|_2)\|\beta^n\|_0\\
\le&\frac{Ch^4}{\tau}\int_{t_{n-1}}^{t_n}\|\mathbf{E}_t\|_2^2dt+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{8t_0}\|\beta^n\|_0,
\end{aligned}\tag{3.11a}$$

$$A_2\le C\|R_1^n\|_0^2+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{8t_0}\|\beta^n\|_0\le C\tau\int_{t_{n-1}}^{t_n}\|\mathbf{E}_{tt}(t)\|_0^2dt+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{8t_0}\|\beta^n\|_0^2,\tag{3.11b}$$

$$A_3\le\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(Ch^2\|\mathbf{E}^n\|_2\|\beta^n\|_0)\le Ch^4\|\mathbf{E}^n\|_2^2+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{8t_0}\|\beta^n\|_0^2,\tag{3.11c}$$

$$A_4=A_7=A_9=0,\tag{3.11d}$$

$$A_5 \le Ch^2 \|\mathbf{P}^n\|_2 \|\beta^n\|_0 \le Ch^4 \|\mathbf{P}^n\|_2^2 + \frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{8t_0}\|\beta^n\|_0^2, \tag{3.11e}$$

$$A_6 \le C\|\gamma^n\|_0^2 + \frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{8t_0}\|\beta^n\|_0^2, \tag{3.11f}$$

$$A_8 \le C\|R_2^n\|_0^2 + C\|\eta^n\|_0^2 \le C\tau\int_{t_{n-1}}^{t_n}\|H_{tt}(t)\|_0^2 dt + C\|\eta^n\|_0^2, \tag{3.11g}$$

$$A_{12} \le C\|u^n-u^{n-1}\|_0\|\mathbf{E}^n\|_{0,\infty}\|\beta^n\|_0 \le C\tau\int_{t_{n-1}}^{t_n}\|u_t(t)\|_0^2 dt + \frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{8t_0}\|\beta^n\|_0^2, \tag{3.11h}$$

where the estimates A_3, A_4, A_5, A_7, A_9 should employ Lemma 2.3.

For any $\omega\in W^{1,\infty}(K)$, define $\overline{\omega}:=\frac{1}{K}\int_K \omega dxdy$ (Lemma 3.4 in [2]), then there holds

$$\|\omega-\overline{\omega}\|_{0,\infty,K} \le Ch_K|\omega|_{1,\infty,K}.$$

Using Lemma 2.4, A_{10} can be estimate as

$$\begin{aligned}
A_{10} \le& \Big|\sum_K\int_K(\sigma(u^{n-1})-\overline{\sigma(u^{n-1})})\alpha^n\beta^n dxdy\Big| + \Big|\sum_K\int_K\overline{\sigma(u^{n-1})}\alpha^n\beta^n dxdy\Big| \\
\le& \sum_K\|\sigma(u^{n-1})-\overline{\sigma(u^{n-1})}\|_{0,\infty,K}\|\alpha^n\|_{0,K}\|\beta^n\|_{0,K} + \sum_K\Big|\int_K\overline{\sigma(u^{n-1})}\alpha^n\beta^n dxdy\Big| \\
\le& Ch^2\sum_K\|\mathbf{E}^n\|_{1,K}\|\beta^n\|_{0,K} + Ch^2\sum_K\|\mathbf{E}^n\|_{2,K}\|\beta^n\|_{0,K} \le Ch^2\sum_K\|\mathbf{E}^n\|_{2,K}\|\beta^n\|_{0,K} \\
\le& Ch^2\|\mathbf{E}^n\|_2\|\beta^n\|_0 \le Ch^4\|\mathbf{E}^n\|_2^2 + \frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{8t_0}\|\beta^n\|_0^2.
\end{aligned} \tag{3.12}$$

To estimate A_{11}, similar to the technique used in [28], we give the following mathematical induction hypothesis.

$$\|\mathbf{E}_h^n\|_{0,\infty} \le \widetilde{C}, \qquad \forall n=0,1,\cdots,M. \tag{3.13}$$

Then, using (3.13), A_{11} can be estimated by

$$\begin{aligned}
A_{11} \le& C\|u^{n-1}-u_h^{n-1}\|_0\|\mathbf{E}_h^n\|_{0,\infty}\|\beta^n\|_0 \le C(h^2\|u^{n-1}\|_2 + \|\zeta^{n-1}\|_0)\|\beta^n\|_0 \\
\le& Ch^4\|u^{n-1}\|_2^2 + C\|\zeta^{n-1}\|_0^2 + \frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{8t_0}\|\beta^n\|_0^2.
\end{aligned} \tag{3.14}$$

Taking the above estimates of A_i, $i=1,2,\cdots,12$ into (3.10) and multiplying by 2τ and summing up from $1\le n\le M$, notice that $\eta^0=0$, $\beta^0=0$, we have

$$\begin{aligned}
&\|\beta^M\|_0^2 + (\mu-C\tau)\|\eta^M\|_0^2 \\
\le& C(h^4+\tau^2) + C\tau\sum_{n=1}^{M-1}\|\eta^n\|_0^2 + C\tau\sum_{n=1}^{M}\|\gamma^n\|_0^2 + C\tau\sum_{n=1}^{M-1}\|\zeta^n\|_0^2.
\end{aligned} \tag{3.15}$$

To get the estimate of $\|\gamma^n\|_0^2$, setting $w_h=\gamma^n$ in (3.9c), we have

$$\begin{aligned}&\Big(\frac{\gamma^n-\gamma^{n-1}}{\tau},\gamma^n\Big)+\frac{1}{t_0}(\gamma^n,\gamma^n)\\ =&-\Big(\frac{\theta^n-\theta^{n-1}}{\tau},\gamma^n\Big)+(R_3^n,\gamma^n)-\frac{1}{t_0}(\theta^n,\gamma^n)+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\mathbf{E}^n-\mathbf{E}^{n-1},\gamma^n)\\ &+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\alpha^{n-1},\gamma^n)+\frac{\epsilon_0(\epsilon_s-\epsilon_\infty)}{t_0}(\beta^{n-1},\gamma^n)\triangleq\sum_{i=1}^{6}B_i. \qquad (3.16)\end{aligned}$$

The left-hand side of (3.16) becomes

$$\Big(\frac{\gamma^n-\gamma^{n-1}}{\tau},\gamma^n\Big)\geq\frac{1}{2\tau}(\|\gamma^n\|_0^2-\|\gamma^{n-1}\|_0^2),\qquad \frac{1}{t_0}(\gamma^n,\gamma^n)=\frac{1}{t_0}\|\gamma^n\|_0^2. \qquad (3.17)$$

Now, we need to estimate the terms B_i, $i=1,2,\cdots,6$.

By Schwarz inequality, ϵ-Young inequality and Lemma 2.3,

$$B_1\leq(Ch^2\tau^{-1}\|\mathbf{P}^n\|_2-Ch^2\tau^{-1}\|\mathbf{P}^{n-1}\|_2)\|\gamma^n\|_0\leq Ch^4\tau^{-1}\int_{t_{n-1}}^{t_n}\|\mathbf{P}_t\|_2^2dt+\frac{1}{6t_0}\|\gamma^n\|_0, \qquad (3.18a)$$

$$B_2\leq C\|R_3^n\|_0^2+\frac{1}{6t_0}\|\gamma^n\|_0^2\leq C\tau\int_{t_{n-1}}^{t_n}\|\mathbf{P}_{tt}(t)\|_0^2dt+\frac{1}{6t_0}\|\gamma^n\|_0^2, \qquad (3.18b)$$

$$B_3\leq Ch^2\|\mathbf{P}^n\|_2\|\gamma^n\|_0\leq Ch^4\|\mathbf{P}^n\|_2^2+\frac{1}{6t_0}\|\gamma^n\|_0^2, \qquad (3.18c)$$

$$B_4\leq C\|\mathbf{E}^n-\mathbf{E}^{n-1}\|_0\|\gamma^n\|_0\leq C\tau\int_{t_{n-1}}^{t_n}\|\mathbf{E}_t(t)\|_0^2dt+\frac{1}{6t_0}\|\gamma^n\|_0^2, \qquad (3.18d)$$

$$B_5\leq Ch^2\|\mathbf{E}^{n-1}\|_2\|\gamma^n\|_0\leq Ch^4\|\mathbf{E}^{n-1}\|_2^2+\frac{1}{6t_0}\|\gamma^n\|_0^2, \qquad (3.18e)$$

$$B_6\leq C\|\beta^{n-1}\|_0\|\gamma^n\|_0\leq C\|\beta^{n-1}\|_0^2+\frac{1}{6t_0}\|\gamma^n\|_0^2, \qquad (3.18f)$$

where the estimates B_1, B_3, B_5 should employ Lemma 2.3.

Taking the above estimates of B_i, $i=1,2,\cdots,6$ into (3.16), and multiplying by 2τ and summing up from $1\leq n\leq M$, notice that $\gamma^0=0$, we have

$$\|\gamma^M\|_0^2\leq C(h^4+\tau^2)+C\tau\sum_{n=1}^{M-1}\|\beta^n\|_0^2. \qquad (3.19)$$

To get the estimate of $\|\zeta^n\|_0^2$, let $v_h=\zeta^n$ in (3.9d)

$$\begin{aligned}&\Big(\frac{\zeta^n-\zeta^{n-1}}{\tau},\zeta^n\Big)+(k\nabla\zeta^n,\nabla\zeta^n)\\ =&-\Big(\frac{\lambda^n-\lambda^{n-1}}{\tau},\zeta^n\Big)-(k\nabla\lambda^n,\nabla\zeta^n)\\ &+((\sigma(u^{n-1})-\sigma(u_h^{n-1}))|\mathbf{E}_h^n|^2,\zeta^n)+(\sigma(u^{n-1})(\mathbf{E}^n+\mathbf{E}_h^n)(\mathbf{E}^n-\mathbf{E}_h^n),\zeta^n)\\ &+((\sigma(u^n)-\sigma(u^{n-1}))|\mathbf{E}^n|^2,\zeta^n)+(R_4^n,\zeta^n)\triangleq\sum_{i=1}^{6}D_i. \qquad (3.20)\end{aligned}$$

The left-hand side of (3.20) becomes

$$\Big(\frac{\zeta^n-\zeta^{n-1}}{\tau},\zeta^n\Big)+(k\nabla\zeta^n,\nabla\zeta^n)\geq\frac{1}{2\tau}(\|\zeta^n\|_0^2-\|\zeta^{n-1}\|_0^2)+k|\zeta^n|_1^2. \tag{3.21}$$

Now, we need to estimate the terms D_i, $i=1,2,\cdots,6$. In fact, it is easy to check that

$$D_1\leq Ch^2\tau^{-1}\int_{t_{n-1}}^{t_n}\|u_t\|_2ds\|\zeta^n\|_0\leq Ch^4\tau^{-1}\int_{t_{n-1}}^{t_n}\|u_t\|_2^2ds+C\|\zeta^n\|_0^2. \tag{3.22}$$

Then according to (2.2) of Lemma 2.2, there holds

$$D_2\leq Ch^2\|u^n\|_4\|\zeta^n\|_0\leq Ch^4\|u^n\|_4^2+C\|\zeta^n\|_0^2. \tag{3.23}$$

By (3.13), D_3 can be estimated as

$$\begin{aligned}D_3\leq&C\|u^{n-1}-u_h^{n-1}\|_0\|\mathbf{E}_h^n\|_{0,\infty}^2\|\zeta^n\|_0\leq C(h^2\|u^{n-1}\|_2+\|\zeta^{n-1}\|_0)\|\zeta^n\|_0\\ \leq&Ch^4\|u^{n-1}\|_2^2+C\|\zeta^{n-1}\|_0^2+C\|\zeta^n\|_0^2.\end{aligned} \tag{3.24}$$

As for D_4, we rewrite it as

$$D_4=(\sigma(u^{n-1})(\mathbf{E}^n+\mathbf{E}_h^n)\cdot\alpha^n,\zeta^n)+(\sigma(u^{n-1})(\mathbf{E}^n+\mathbf{E}_h^n)\cdot\beta^n,\zeta^n)=G_1+G_2.$$

Furthermore, G_1 can be written as

$$\begin{aligned}G_1=&2(\sigma(u^{n-1})\mathbf{E}^n\cdot\alpha^n,\zeta^n)+(\sigma(u^{n-1})(\mathbf{E}_h^n-\mathbf{E}^n)\cdot\alpha^n,\zeta^n)\\ =&2(\sigma(u^{n-1})E_1^n\cdot\alpha_1^n,\zeta^n)+2(\sigma(u^{n-1})E_2^n\cdot\alpha_2^n,\zeta^n)+(\sigma(u^{n-1})(\mathbf{E}_h^n-\mathbf{E}^n)\cdot\alpha^n,\zeta^n)\\ =&J_1+J_2+J_3.\end{aligned}$$

For J_1, due to Lemmas 2.1, 2.4 and $H^1(\Omega)\hookrightarrow L^4(\Omega)$ in two-dimensional space, we have

$$\begin{aligned}J_1=&2\sum_K\int_K[(\sigma(u^{n-1})-\overline{\sigma(u^{n-1})})E_1^n\alpha_1^n\zeta^n+\overline{\sigma(u^{n-1})}(E_1^n-\overline{E_1^n})\alpha_1^n\zeta^n\\ &\quad+\overline{\sigma(u^{n-1})}\ \overline{E_1^n}\alpha_1^n\zeta^n]dxdy\\ \leq&C\sum_K\|\sigma(u^{n-1})-\overline{\sigma(u^{n-1})}\|_{0,\infty,K}\|E_1^n\|_{0,\infty,K}\|\alpha_1^n\|_{0,K}\|\zeta^n\|_{0,K}\\ &\quad+C\sum_K\|E_1^n-\overline{E_1^n}\|_{0,2}\|\alpha_1^n\|_{0,4}\|\zeta^n\|_{0,4}+C\sum_K h_K^2\|E_1^n\|_{2,K}|\zeta^n|_{1,K}\\ \leq&Ch^2\sum_K\|E_1^n\|_{1,K}\|\zeta^n\|_{0,K}+Ch^2\sum_K\|E_1^n\|_{1,2}\|E_1^n\|_{1,4}|\zeta^n|_{1,K}+Ch^2\sum_K\|E_1^n\|_{2,K}|\zeta^n|_{1,K}\\ \leq&Ch^2\|E_1^n\|_{2,K}|\zeta^n|_{1,K}\leq Ch^4\|E_1^n\|_2^2+\frac{k}{3}|\zeta^n|_1^2.\end{aligned} \tag{3.25}$$

Similarly, we have

$$J_2\leq Ch^4\|E_2^n\|_2^2+\frac{k}{3}|\zeta^n|_1^2. \tag{3.26}$$

Furthermore

$$\begin{aligned}J_3 \leq & C\|\mathbf{E}_h^n-\mathbf{\Pi}_h\mathbf{E}^n\|_{0,2}\|\alpha^n\|_{0,4}\|\zeta^n\|_{0,4}+C\|\mathbf{\Pi}_h\mathbf{E}^n-\mathbf{E}^n\|_{0,4}\|\alpha^n\|_{0,4}\|\zeta^n\|_{0,2}\\ \leq & C\|\beta^n\|_{0,2}\|\alpha^n\|_{0,4}\|\zeta^n\|_{0,4}+Ch^2\|\mathbf{E}^n\|_{1,4}^2\|\zeta^n\|_0\\ \leq & C\|\beta^n\|_{0,2}\|\alpha^n\|_{0,4}\|\zeta^n\|_1+Ch^2\|\mathbf{E}^n\|_{1,4}^2\|\zeta^n\|_0\\ \leq & Ch\|\beta^n\|_{0,2}\|\mathbf{E}^n\|_{1,4}\|\zeta^n\|_1+Ch^2\|\mathbf{E}^n\|_{1,4}^2\|\zeta^n\|_0\\ \leq & Ch^4+C\|\beta^n\|_0^2+\frac{k}{3}|\zeta^n|_1^2. \end{aligned} \tag{3.27}$$

For G_2, we have

$$G_2 \leq C\|\sigma(u^{n-1})\|_{0,\infty}\|\mathbf{E}^n+\mathbf{E}_h^n\|_{0,\infty}\|\beta^n\|_0\|\zeta^n\|_0 \leq C\|\beta^n\|_0^2+C\|\zeta^n\|_0^2, \tag{3.28a}$$

$$\begin{aligned}D_5+D_6 \leq & C\|u^n-u^{n-1}\|_0\|\mathbf{E}^n\|_{0,\infty}^2\|\zeta^n\|_0+C\|R_4^n\|_0\|\zeta^n\|_0\\ \leq & C\tau\int_{t_{n-1}}^{t_n}\|u_t(t)\|_0^2dt+C\tau\int_{t_{n-1}}^{t_n}\|u_{tt}(t)\|_0^2dt+C\|\zeta^n\|_0^2.\end{aligned} \tag{3.28b}$$

Then taking (3.21)-(3.28b) into (3.20), we derive

$$\begin{aligned}&\frac{1}{2\tau}(\|\zeta^n\|_0^2-\|\zeta^{n-1}\|_0^2)\\ \leq & Ch^4\tau^{-1}\int_{t_{n-1}}^{t_n}\|u_t\|_2^2ds+C(h^4+\tau^2)+C\|\zeta^n\|_0^2+C\|\zeta^{n-1}\|_0^2+C\|\beta^n\|_0^2.\end{aligned}$$

Multiplying by 2τ and summing up from $n=1,\cdots,M$, $(1\leq M\leq N)$, we obtain

$$(1-C\tau)\|\zeta^M\|_0^2 \leq C(h^4+\tau^2)+C\tau\sum_{n=1}^{M-1}\|\zeta^n\|_0^2+C\tau\sum_{n=1}^{M}\|\beta^n\|_0^2. \tag{3.29}$$

By (3.15), (3.19) and (3.29), choosing proper ϵ_0, μ, τ, so that $\epsilon_0-C\tau$, $\mu-C\tau$, $1-C\tau$ are greater than 0, and applying discrete Grönwall inequality, we get

$$\|\beta^M\|_0+\|\eta^M\|_0+\|\gamma^M\|_0+\|\zeta^M\|_0 \leq C(\tau+h^2). \tag{3.30}$$

Now, let's verify (3.13). First, when $n=0$,

$$\begin{aligned}\|\mathbf{E}_h^0\|_{0,\infty}=&\|\mathbf{\Pi}_h\mathbf{E}(0)\|_{0,\infty} \leq \|\mathbf{\Pi}_h\mathbf{E}(0)-\mathbf{E}(0)\|_{0,\infty}+\|\mathbf{E}(0)\|_{0,\infty}\\ \leq & Ch\|\mathbf{E}(0)\|_{1,\infty}+\|\mathbf{E}(0)\|_{0,\infty} \leq \widetilde{C}.\end{aligned} \tag{3.31}$$

Assume that (3.13) holds true for $n=k-1$, then there holds from (3.30) that

$$\|\mathbf{\Pi}_h\mathbf{E}^{k-1}-\mathbf{E}_h^{k-1}\|_0 \leq C(h^2+\tau).$$

Then when $n=k$, we need to verify that $\|\mathbf{E}_h^k\|_{0,\infty}<\widetilde{C}$. Moreover, we know that $\|\mathbf{E}_h(t)\|_{0,\infty}$ is a continuous function with respect to t, so for a given $\epsilon>0$, there exists $\delta>0$, such that when $|t_k-t_{k-1}|=\tau<\delta$, there holds

$$\big|\|\mathbf{E}_h^k\|_{0,\infty}-\|\mathbf{E}_h^{k-1}\|_{0,\infty}\big|\le\epsilon.$$

Let $\epsilon=h$,

$$\begin{aligned}\|\mathbf{E}_h^k\|_{0,\infty}\le&\|\mathbf{E}_h^{k-1}\|_{0,\infty}+\big|\|\mathbf{E}_h^k\|_{0,\infty}-\|\mathbf{E}_h^{k-1}\|_{0,\infty}\big|\\ \le&\|\mathbf{E}_h^{k-1}-\mathbf{\Pi}_h\mathbf{E}^{k-1}\|_{0,\infty}+\|\mathbf{\Pi}_h\mathbf{E}^{k-1}-\mathbf{E}^{k-1}\|_{0,\infty}+\|\mathbf{E}^{k-1}\|_{0,\infty}+h\\ \le&Ch^{-1}\|\mathbf{E}_h^{k-1}-\mathbf{\Pi}_h\mathbf{E}^{k-1}\|_0+Ch\|\mathbf{E}^{k-1}\|_{1,\infty}+\|\mathbf{E}^{k-1}\|_{0,\infty}+h\\ \le&C(h+h^{-1}\tau)+\|\mathbf{E}\|_{L^\infty(W^{0,\infty}(\Omega))}\le\widetilde{C},\end{aligned}$$

where $\tau=\mathcal{O}(h^{1+\gamma})$, $\gamma>0$ is needed and h is sufficiently small such that $Ch^\gamma<1$. Thus we can choose $\widetilde{C}=1+\|\mathbf{E}\|_{L^\infty(W^{0,\infty}(\Omega))}$. This implies that the mathematical induction (3.13) holds uniformly for any n.

Now, we pay attention to the estimate of $|\zeta^n|_1^2$. Let $v_h=\zeta^n-\zeta^{n-1}$ in (3.9d)

$$\begin{aligned}&\Big(\frac{\zeta^n-\zeta^{n-1}}{\tau},\zeta^n-\zeta^{n-1}\Big)+(k\nabla\zeta^n,\nabla(\zeta^n-\zeta^{n-1}))\\ =&-\Big(\frac{\lambda^n-\lambda^{n-1}}{\tau},\zeta^n-\zeta^{n-1}\Big)-(k\nabla\lambda^n,\nabla(\zeta^n-\zeta^{n-1}))+((\sigma(u^{n-1})-\sigma(u_h^{n-1}))|\mathbf{E}_h^n|^2,\zeta^n-\zeta^{n-1})\\ &+(\sigma(u^{n-1})(\mathbf{E}^n+\mathbf{E}_h^n)(\mathbf{E}^n-\mathbf{E}_h^n),\zeta^n-\zeta^{n-1})+((\sigma(u^n)-\sigma(u^{n-1}))|\mathbf{E}^n|^2,\zeta^n-\zeta^{n-1})\\ &+(R_4^n,\zeta^n-\zeta^{n-1})\triangleq\sum_{i=1}^6F_i.\end{aligned}\tag{3.32}$$

The left-hand side of (3.32) becomes

$$\begin{aligned}&(\zeta^n-\zeta^{n-1}\tau,\zeta^n-\zeta^{n-1})+(k\nabla\zeta^n,\nabla(\zeta^n-\zeta^{n-1}))\\ \ge&\frac{1}{\tau}(\|\zeta^n-\zeta^{n-1}\|_0^2+\frac{k}{2}(|\zeta^n|_1^2-|\zeta^{n-1}|_1^2).\end{aligned}\tag{3.33}$$

Now, we start to estimate each term F_i, $i=1,2,\cdots,6$. First, in the same way as D_1, D_2 and D_3, we can check that

$$F_1\le Ch^2\tau^{-1}\int_{t_{n-1}}^{t_n}\|u_t\|_2ds\|\zeta^n-\zeta^{n-1}\|_0\le Ch^4\int_{t_{n-1}}^{t_n}\|u_t\|_2^2ds+\frac{1}{6\tau}\|\zeta^n-\zeta^{n-1}\|_0^2,\tag{3.34}$$

$$F_2\le Ch^2\|u^n\|_4\|\zeta^n-\zeta^{n-1}\|_0\le C\tau h^4+\frac{1}{6\tau}\|\zeta^n-\zeta^{n-1}\|_0^2,\tag{3.35}$$

$$\begin{aligned}F_3\le&C\|u^{n-1}-u_h^{n-1}\|_0\|\mathbf{E}_h^n\|_{0,\infty}^2\|\zeta^n-\zeta^{n-1}\|_0\le C(h^2\|u^{n-1}\|_0+\|\zeta^{n-1}\|_0)\|\zeta^n-\zeta^{n-1}\|_0\\ \le&C(h^2+\tau)\|\zeta^n-\zeta^{n-1}\|_0\le C\tau(h^4+\tau^2)+\frac{1}{6\tau}\|\zeta^n-\zeta^{n-1}\|_0^2.\end{aligned}\tag{3.36}$$

Second, we rewrite F_4 as

$$\begin{aligned}
F_4 =&(\sigma(u^{n-1})(\mathbf{E}^n+\mathbf{E}_h^n)(\mathbf{E}^n-\mathbf{E}_h^n),\zeta^n-\zeta^{n-1})\\
=&(\sigma(u^{n-1})(\mathbf{E}^n+\mathbf{E}_h^n)\cdot\alpha^n,\zeta^n-\zeta^{n-1})+(\sigma(u^{n-1})(\mathbf{E}^n+\mathbf{E}_h^n)\cdot\beta^n,\zeta^n-\zeta^{n-1})\\
=&2(\sigma(u^{n-1})\mathbf{E}^n\cdot\alpha^n),\zeta^n-\zeta^{n-1})+(\sigma(u^{n-1})(\mathbf{E}_h^n-\mathbf{E}^n)\cdot\alpha^n,\zeta^n-\zeta^{n-1})\\
&+(\sigma(u^{n-1})(\mathbf{E}^n+\mathbf{E}_h^n)\cdot\beta^n,\zeta^n-\zeta^{n-1})\\
=&2(\sigma(u^{n-1})E_1^n\cdot\alpha_1^n),\zeta^n-\zeta^{n-1})+2(\sigma(u^{n-1})E_2^n\cdot\alpha_2^n),\zeta^n-\zeta^{n-1})\\
&+(\sigma(u^{n-1})(\mathbf{E}_h^n-\mathbf{E}^n)\cdot\alpha^n,\zeta^n-\zeta^{n-1})+(\sigma(u^{n-1})(\mathbf{E}^n+\mathbf{E}_h^n)\cdot\beta^n,\zeta^n-\zeta^{n-1})\\
=&F_{41}+F_{42}+F_{43}+F_{44}.
\end{aligned}$$

Now, we start to estimate each term F_{4i}, $(i=1,2,3,4)$. Due to Lemma 2.1 and Lemma 2.3-2.4, inverse inequality, we have

$$\begin{aligned}
F_{43}\le& C\|\mathbf{E}_h^n-\mathbf{\Pi}_h\mathbf{E}^n\|_{0,4}\|\alpha^n\|_{0,4}\|\zeta^n-\zeta^{n-1}\|_{0,K}+C\|\mathbf{\Pi}_h\mathbf{E}^n-\mathbf{E}^n\|_{0,4}\|\alpha^n\|_{0,4}\|\zeta^n\|_{0,K}\\
\le& Ch^{-\frac{1}{2}}(h^2+\tau)Ch\|\mathbf{E}^n\|_{1,4}\|\zeta^n-\zeta^{n-1}\|_{0,K}+Ch^2\|\mathbf{E}^n\|_{1,4}^2\|\zeta^n-\zeta^{n-1}\|_0\\
\le& C(h^{\frac{5}{2}}+h^{\frac{1}{2}}\tau)\|\zeta^n-\zeta^{n-1}\|_{0,K}+Ch^2\|\zeta^n-\zeta^{n-1}\|_0\\
\le& C\tau(h^4+\tau^2)+\frac{1}{6\tau}\|\zeta^n-\zeta^{n-1}\|_0^2,\\
F_{44}\le& C\|\sigma(u^{n-1})\|_{0,\infty}\|\mathbf{E}^n+\mathbf{E}_h^n\|_{0,\infty}\|\beta^n\|_0\|\zeta^n-\zeta^{n-1}\|_0\le C(h^2+\tau)\|\zeta^n-\zeta^{n-1}\|_0\\
\le& C\tau(h^4+\tau^2)+\frac{1}{6\tau}\|\zeta^n-\zeta^{n-1}\|_0^2.
\end{aligned}$$

For F_{41}, using the identity $(a^n,b^n-b^{n-1})=(a^n,b^n)-(a^{n-1},b^{n-1})-(a^n-a^{n-1},b^{n-1})$, we have

$$\begin{aligned}
F_{41}=&2[(\sigma(u^{n-1})E_1^n\cdot\alpha_1^n,\zeta^n)-(\sigma(u^{n-2})E_1^{n-1}\cdot\alpha_1^{n-1},\zeta^{n-1})]\\
&-2(\sigma(u^{n-1})E_1^n\cdot\alpha_1^n-\sigma(u^{n-2})E_1^{n-1}\cdot\alpha_1^{n-1},\zeta^{n-1})\\
=&2[(\sigma(u^{n-1})E_1^n\cdot\alpha_1^n,\zeta^n)-(\sigma(u^{n-2})E_1^{n-1}\cdot\alpha_1^{n-1},\zeta^{n-1})]\\
&-2[((\sigma(u^{n-1})-\sigma(u^{n-2}))E_1^n\cdot\alpha_1^n,\zeta^{n-1})+(\sigma(u^{n-2})(E_1^n-E_1^{n-1})\alpha_1^n,\zeta^{n-1})\\
&+(\sigma(u^{n-2})E_1^{n-1}(\alpha_1^n-\alpha_1^{n-1}),\zeta^{n-1})].
\end{aligned}$$

For the third term of the above equation, applying the mean-value inequality, we have

$$\begin{aligned}
&-2((\sigma(u^{n-1})-\sigma(u^{n-2}))E_1^n\cdot\alpha_1^n,\zeta^{n-1})\\
=&-2\sum_K\int_K[(\sigma'(\lambda)-\overline{\sigma'(\lambda)})(u^{k-1}-u^{k-2})E_1^n\alpha_1^n\zeta^{n-1}\\
&+\overline{\sigma'(\lambda)}(u^{k-1}-u^{k-2}-\overline{u^{k-1}-u^{k-2}})E_1^n\alpha_1^n\zeta^{n-1}\\
&+\overline{\sigma'(\lambda)}\,\overline{u^{k-1}-u^{k-2}}(E_1^n-\overline{E_1^n})\alpha_1^n\zeta^{n-1}+\overline{\sigma'(\lambda)}\,\overline{u^{k-1}-u^{k-2}}\overline{E_1^n}\alpha_1^n\zeta^{n-1}]dxdy\\
\le&\sum_K C_Kh_K^2\tau\|E_1^n\|_{1,K}\|\zeta^{n-1}\|_{0,K}+\sum_K C_Kh_K^2\tau\|E_1^n\|_{1,K}\|\zeta^{n-1}\|_{0,K}\\
&+\sum_K C_Kh_K^2\tau\|E_1^n\|_{1,K}\|E_1^n\|_{1,4}\|\zeta^{n-1}\|_{0,4}+\sum_K C_Kh_K^2\tau\|E_1^n\|_{2,K}\|\zeta^{n-1}\|_{1,K}
\end{aligned}$$

$$\leq Ch^2\tau|\zeta^{n-1}|_1 \leq Ch^4\tau + C\tau|\zeta^n|_1^2,$$

where λ is between at u^{n-1} and u^{n-2}.

Similarly, we can derive

$$\begin{aligned}
&-2(\sigma(u^{n-2})(E_1^n - E_1^{n-1})\alpha_1^n, \zeta^{n-1}) \leq Ch^4\tau + C\tau|\zeta^n|_1^2,\\
&-2(\sigma(u^{n-2})E_1^{n-1}(\alpha_1^n - \alpha_1^{n-1}), \zeta^{n-1}) \leq Ch^4\tau + C\tau|\zeta^n|_1^2.
\end{aligned}$$

Therefore, F_{41} reduce to

$$F_{41} \leq 2[(\sigma(u^{n-1})E_1^n \cdot \alpha_1^n, \zeta^n) - (\sigma(u^{n-2})E_1^{n-1} \cdot \alpha_1^{n-1}, \zeta^{n-1})] + Ch^4\tau + C\tau|\zeta^n|_1^2.$$

In the same way, we have

$$F_{42} \leq 2[(\sigma(u^{n-1})E_2^n \cdot \alpha_2^n, \zeta^n) - (\sigma(u^{n-2})E_2^{n-1} \cdot \alpha_2^{n-1}, \zeta^{n-1})] + Ch^4\tau + C\tau|\zeta^n|_1^2.$$

Furthermore, it follows that

$$\begin{aligned}
F_4 \leq & 2[(\sigma(u^{n-1})E_1^n \cdot \alpha_1^n, \zeta^n) - (\sigma(u^{n-2})E_1^{n-1} \cdot \alpha_1^{n-1}, \zeta^{n-1})]\\
& + 2[(\sigma(u^{n-1})E_2^n \cdot \alpha_2^n, \zeta^n) - (\sigma(u^{n-2})E_2^{n-1} \cdot \alpha_2^{n-1}, \zeta^{n-1})]\\
& + C(h^4 + \tau^2)\tau + C\tau|\zeta^n|_1^2 + \frac{1}{3\tau}\|\zeta^n - \zeta^{n-1}\|_0^2. \qquad (3.37)
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
F_5 + F_6 \leq & C\|u^n - u^{n-1}\|_0 \|\mathbf{E}^n\|_{0,\infty}^2 \|\zeta^n - \zeta^{n-1}\|_0 + C\|R_4^n\|_0^2 \|\zeta^n - \zeta^{n-1}\|_0\\
\leq & C\tau^2 \int_{t_{n-1}}^{t_n} \|u_t(t)\|_0^2 dt + C\tau^2 \int_{t_{n-1}}^{t_n} \|u_{tt}(t)\|_0^2 dt + \frac{1}{6\tau}\|\zeta^n - \zeta^{n-1}\|_0^2. \qquad (3.38)
\end{aligned}$$

Then taking (3.33)-(3.38) into (3.32), and summing up the above inequality and noticing that $\zeta^0 = 0$, we obtain

$$\begin{aligned}
&\left(\frac{k}{2} - C\tau\right)|\zeta^M|_1^2\\
\leq & 2(\sigma(u^{M-1})E_1^M \cdot \alpha_1^M, \zeta^M) + 2(\sigma(u^{M-1})E_2^M \cdot \alpha_2^M, \zeta^M) + C(h^4 + \tau^2) + C\tau \sum_{n=1}^{M-1} |\zeta^n|_1^2.
\end{aligned}$$

By choosing proper τ so that $\frac{k}{2} - C\tau > 0$, applying the similar process as J_1 for the first and second terms in the right-hand side of the above inequality, we have

$$|\zeta^M|_1^2 \leq C(h^4 + \tau^2) + C\tau \sum_{n=1}^{M-1} |\zeta^n|_1^2,$$

by applying discrete Grönwall inequality, we can complete the proof. □

4 Global superconvergence analysis

To obtain global superconvergence, we merge the adjacent four elements K_1, K_2, K_3, K_4 into one big element $\widetilde{K}=\bigcup_{i=1}^4 K_i$ (Fig. 1) without overlapped.

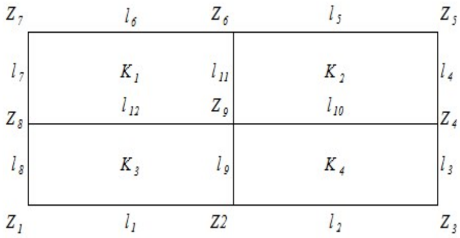

Figure 1: Large unit $\widetilde{K}$.

By use of the postprocessing interpolation operators $\mathbf{\Pi}_{2h}^2$, $\mathbf{J}_{2h}^2$, $\mathbf{I}_{2h}^2$ constructed in [14]:

$$\mathbf{\Pi}_{2h}^2\mathbf{w}|_{\widetilde{K}}\to Q_{11}(\widetilde{K})\times Q_{11}(\widetilde{K}),\quad \forall\mathbf{w}=(w_1,w_2)\in(H^2(\widetilde{K}))^2,\quad \widetilde{K}\in\Gamma_{2h},$$
$$\int_{l_i}(\mathbf{\Pi}_{2h}^2 w_1-w_1)ds=0,\quad i=1,2,5,6;\quad \int_{l_i}(\mathbf{\Pi}_{2h}^2 w_2-w_2)ds=0,\quad i=3,4,7,8;$$
$$\mathbf{J}_{2h}^2 q|_{\widetilde{K}}\in Q_{11},\quad \int_{e_i}(\mathbf{J}_{2h}^2 q-q)dxdy=0,\quad i=1,2,3,4;$$
$$\mathbf{I}_{2h}^2 u|_{\widetilde{K}}\in Q_{22}(\widetilde{K}),\quad \forall u\in C(\widetilde{K}),\quad \mathbf{I}_{2h}^2 u(Z_i)=u(Z_i),\quad i=1,\cdots,9,$$

where $C(\widetilde{K})$ is a continuous function space on $\widetilde{K}$, the postprocessing interpolation operators $\mathbf{\Pi}_{2h}^2$, $\mathbf{J}_{2h}^2$, $\mathbf{I}_{2h}^2$ (details in [14]) satisfy

$$\mathbf{\Pi}_{2h}^2\mathbf{\Pi}_h\mathbf{w}=\mathbf{\Pi}_{2h}^2\mathbf{w},\quad \mathbf{J}_{2h}^2\mathbf{J}_h q=\mathbf{J}_{2h}^2 q,\quad \mathbf{I}_{2h}^2\mathbf{I}_h u=\mathbf{I}_{2h}^2 u, \tag{4.1a}$$
$$\|\mathbf{\Pi}_{2h}^2\mathbf{w}-\mathbf{w}\|_0\le Ch^2\|\mathbf{w}\|_2,\quad \forall\mathbf{w}\in(H^2(\Omega))^2, \tag{4.1b}$$
$$\|\mathbf{J}_{2h}^2 q-q\|_0\le Ch^2\|q\|_2,\quad \forall q\in H^2(\Omega),\quad \|\mathbf{I}_{2h}^2 u\|_1\le Ch^2\|u\|_3,\quad \forall u\in H^3(\Omega), \tag{4.1c}$$
$$\|\mathbf{\Pi}_{2h}^2\mathbf{w}_h\|_0\le C\|\mathbf{w}_h\|_0,\quad \forall\mathbf{w}_h\in\mathbf{N}_h,\quad \|\mathbf{J}_{2h}^2 q_h\|_0\le C\|q_h\|_0,\quad \forall q_h\in W_h, \tag{4.1d}$$
$$\|\mathbf{I}_{2h}^2 v_h\|_1\le C\|v_h\|_1,\quad \forall v_h\in V_0^h. \tag{4.1e}$$

Using these post-processing operators, we can achieve the following global superconvergence for all three dispersive media.

Theorem 4.1. *Under the conditions of Theorem 3.2, there holds the following global superconvergent results*

$$\begin{aligned}&\max_{1\le n\le N}\big(\|\mathbf{E}^n-\mathbf{\Pi}_{2h}^2\mathbf{E}_h^n\|_0+\|\mathbf{P}^n-\mathbf{\Pi}_{2h}^2\mathbf{P}_h^n\|_0\\&\qquad+\|H^n-\mathbf{J}_{2h}^2 H_h^n\|_0+\|\nabla(u^n-\mathbf{I}_{2h}^2 u_h^n)\|_0\big)\\&\le C(h^2+\tau).\end{aligned}$$

Proof. Notice that

$$\mathbf{E}^n - \mathbf{\Pi}_{2h}^2 \mathbf{E}_h^n = \mathbf{E}^n - \mathbf{\Pi}_{2h}^2 \mathbf{\Pi}_h \mathbf{E}^n + \mathbf{\Pi}_{2h}^2 \mathbf{\Pi}_h \mathbf{E}^n - \mathbf{\Pi}_{2h}^2 \mathbf{E}_h^n, \tag{4.2}$$

using (4.1a)-(4.1e), we have

$$\|\mathbf{E}^n - \mathbf{\Pi}_{2h}^2 \mathbf{\Pi}_h \mathbf{E}^n\|_0 = \|\mathbf{E}^n - \mathbf{\Pi}_{2h}^2 \mathbf{E}^n\|_0 \le Ch^2 \|\mathbf{E}^n\|_2, \tag{4.3}$$

and

$$\|\mathbf{\Pi}_{2h}^2 \mathbf{\Pi}_h \mathbf{E}^n - \mathbf{\Pi}_{2h}^2 \mathbf{E}_h^n\|_0 = \|\mathbf{\Pi}_{2h}^2 (\mathbf{\Pi}_h \mathbf{E}^n - \mathbf{E}_h^n)\|_0 \le C\|\mathbf{\Pi}_h \mathbf{E}^n - \mathbf{E}_h^n\|_0 \le C(h^2+\tau). \tag{4.4}$$

According to (4.2)-(4.4), and using triangle inequality,

$$\|\mathbf{E}^n - \mathbf{\Pi}_{2h}^2 \mathbf{E}_h^n\|_0 \le \|\mathbf{E}^n - \mathbf{\Pi}_{2h}^2 \mathbf{\Pi}_h \mathbf{E}^n\|_0 + \|\mathbf{\Pi}_{2h}^2 \mathbf{\Pi}_h \mathbf{E}^n - \mathbf{\Pi}_{2h}^2 \mathbf{E}_h^n\|_0 \le C(h^2+\tau).$$

Similarly, we have

$$\begin{aligned}
&\|\mathbf{P}^n - \mathbf{\Pi}_{2h}^2 \mathbf{P}_h^n\|_0 \le C(h^2+\tau),\\
&\|H^n - \mathbf{J}_{2h}^2 H_h^n\|_0 \le C(h^2+\tau),\\
&\|\nabla(u^n - \mathbf{I}_{2h}^2 u_h^n)\|_0) \le C(h^2+\tau).
\end{aligned}$$

The proof can be completed. □

5 Numerical examples

In this section, we provide a numerical examples to confirm the theoretical analysis. Let the domain $\Omega=[0,1]\times[0,1]$, and the exact solutions

$$\begin{aligned}
&\mathbf{E}(x,y,t) = [-e^{-t}\cos(\pi x)\sin(\pi y), e^{-t}\sin(\pi x)\cos(\pi y)],\\
&H(x,y,t) = 2\pi e^{-t}(\cos(\pi x)\cos(\pi y)),\\
&\mathbf{P}(x,y,t) = [2e^{-t}\cos(\pi x)\sin(\pi y), -2e^{-t}\sin(\pi x)\cos(\pi y)],\\
&u(x,y,t) = e^{-t}\sin(\pi x)\sin(\pi y),
\end{aligned}$$

with the electric conductivity

$$\sigma(u) = \frac{1}{1+u^2}+1.$$

We divide the domain Ω into $N\times N$ uniform rectangles, and choose $\tau=O(h^2)$. The convergence and superconvergence results of $\mathbf{E}$, H, $\mathbf{P}$, and u with respect to $t=0.5,1.0$ are listed in Tables 1-8, respectively. It can be seen from Tables 1-4 that $\|\mathbf{E}^n-\mathbf{E}_h^n\|_0$, $\|H^n-H_h^n\|_0$, $\|\mathbf{P}^n-\mathbf{P}_h^n\|_0$, $\|u^n-u_h^n\|_1$ are convergent at optimal rate of $\mathcal{O}(h)$, respectively. From Tables 5-8, we can see that $\|\mathbf{E}^n-\mathbf{\Pi}_{2h}^2\mathbf{E}_h^n\|_0$, $\|H^n-\mathbf{J}_{2h}^2H_h^n\|_0$, $\|\mathbf{P}^n-\mathbf{\Pi}_{2h}^2\mathbf{P}_h^n\|_0$, $\|u_h^n-\mathbf{I}_{2h}^2u_h^n\|_1$ are of scale $\mathcal{O}(h^2)$, respectively. This data coincide with the theoretical analysis. Moreover, for clarity, we also plot the convergence errors and superconvergent errors using logarithm scales in Figs. 2 and 3.

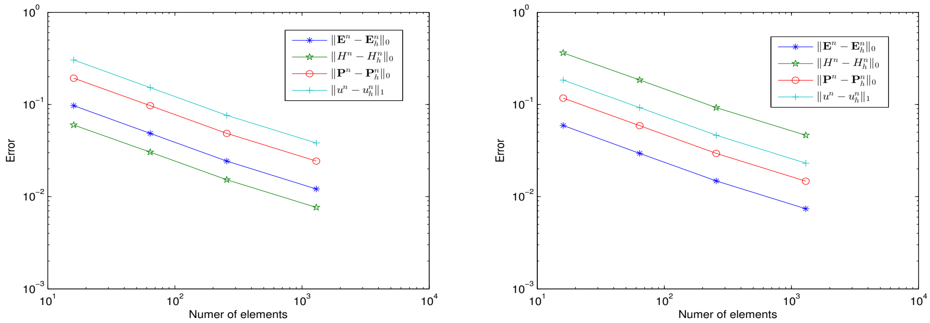

Figure 2: The errors at $t=0.5$(left) and $t=1.0$(right).

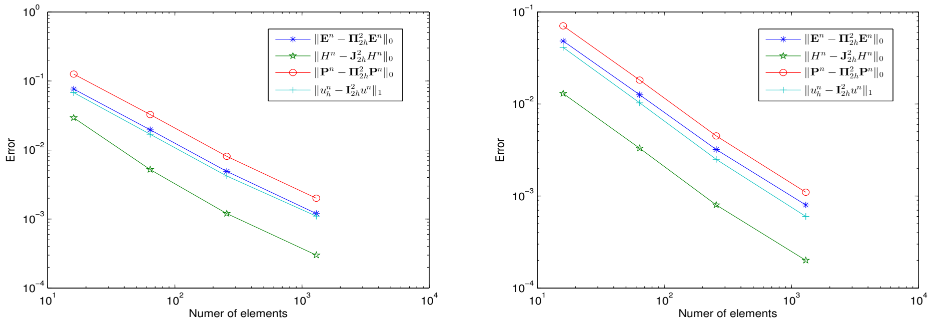

Figure 3: The superconvergent errors at $t=0.5$(Left) and $t=1$(right).

Table 1: Convergent results at $t=0.5$.

$N\times N$	$\lVert\mathbf{E}^n-\mathbf{E}_h^n\rVert_0$	Order	$\lVert H^n-H_h^n\rVert_0$	Order	$\lVert\mathbf{P}^n-\mathbf{P}_h^n\rVert_0$	Order
4×4	0.0971	-	0.5982	-	0.1929	-
8×8	0.0486	0.9981	0.3039	0.9771	0.0970	0.9913
16×16	0.0243	0.9999	0.1525	0.9943	0.0486	0.9977
32×32	0.0121	1.0000	0.0764	0.9986	0.0243	0.9994

Table 2: Convergent results at $t=0.5$.

$N\times N$	$\lVert u^n-u_h^n\rVert_0$	Order	$\lVert u^n-u_h^n\rVert_1$	Order
4×4	0.0160	-	0.3035	-
8×8	0.0041	1.9783	0.1525	0.9913
16×16	0.0010	1.9950	0.0763	0.9981
32×32	0.0003	1.9988	0.0382	0.9995

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Table 3: Convergent results at $t=1.0$.

$N\times N$	$\|\mathbf{E}^n-\mathbf{E}_h^n\|_0$	Order	$\|H^n-H_h^n\|_0$	Order	$\|\mathbf{P}^n-\mathbf{P}_h^n\|_0$	Order
4×4	0.0592	-	0.3627	-	0.1172	-
8×8	0.0295	1.0033	0.1843	0.9769	0.0589	0.9927
16×16	0.0148	1.0020	0.0925	0.9942	0.0295	0.9983
32×32	0.0074	1.0006	0.0463	0.9985	0.0147	0.9996

Table 4: Convergent results at $t=1.0$.

$N\times N$	$\|u^n-u_h^n\|_0$	Order	$\|u^n-u_h^n\|_1$	Order
4×4	0.0094	-	0.1841	-
8×8	0.0023	2.0539	0.0924	0.9930
16×16	0.0005	2.0067	0.0463	0.9981
32×32	0.0001	2.0010	0.0231	0.9995

Table 5: Super-convergent phenomenon at $t=0.5$.

$N\times N$	$\|\mathbf{E}^n-\mathbf{\Pi}_{2h}^2\mathbf{E}_h^n\|_0$	Order	$\|H^n-\mathbf{J}_{2h}^2H_h^n\|_0$	Order
4×4	0.0765	-	0.0293	-
8×8	0.0196	1.9664	0.0052	2.2035
16×16	0.0049	1.9903	0.0011	2.0621
32×32	0.0012	1.9976	0.0003	2.0164

Table 6: Super-convergent phenomenon at $t=0.5$.

$N\times N$	$\|\mathbf{P}^n-\mathbf{\Pi}_{2h}^2\mathbf{P}_h^n\|_0$	Order	$\|u_h^n-\mathbf{I}_{2h}^2u_h^n\|_1$	Order
4×4	0.1258	-	0.0679	-
8×8	0.0326	1.9964	0.0169	1.9991
16×16	0.0081	2.0004	0.0042	1.9996
32×32	0.0020	2.0002	0.0011	1.9999

Table 7: Super-convergent phenomenon at $t=1.0$.

$N\times N$	$\|\mathbf{E}^n-\mathbf{\Pi}_{2h}^2\mathbf{E}_h^n\|_0$	Order	$\|H^n-\mathbf{J}_{2h}^2H_h^n\|_0$	Order
4×4	0.0483	-	0.0130	-
8×8	0.0126	1.9790	0.0033	2.2562
16×16	0.0032	1.9943	0.0008	2.0632
32×32	0.0008	1.9985	0.0002	2.0172

Table 8: Super-convergent phenomenon at $t=1.0$.

$N\times N$	$\|\mathbf{P}^n-\mathbf{\Pi}_{2h}^2\mathbf{P}_h^n\|_0$	Order	$\|u_h^n-\mathbf{I}_{2h}^2u_h^n\|_1$	Order
4×4	0.0705	-	0.0411	-
8×8	0.0183	2.0046	0.0102	1.9998
16×16	0.0044	2.0012	0.0024	1.9999
32×32	0.0011	2.0010	0.0006	2.0000

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