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Stream Helicity in Toroidal Domains

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Abstract. As the analogue of helicity, the concept of stream helicity proposed in 2007 provides a quantitative measurement for entanglement of streamlines for an incompressible flow field or magnetic field lines for a magnetic field. It is found that the helical-wave decomposition can serve as a convenient mathematical physics tool for the calculation of helicity and stream helicity. However, for a multiply connected domain such as a torus under the background of TOKAMAK, the analysis presents some peculiarities due to the nature of multiple-value of some involving potentials. Here we give a full derivation of stream helicity for such a case, together with some detailed analysis. Particularly, it is found that a maximum of absolute helicity under prescribed energy could be attained by a pure potential field.

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Key words: Helicity, magnetic field, streamlines, toroidal domain, helical-wave decomposition.

1 Introduction

The geometrical description for the structures in a flow or a magnetic field is in the most fundamental position when studying the kinematics or dynamics of the flow or the magnetic field. In this regard, it is well known that the helicity defined as

$$H \equiv \int \boldsymbol{\omega} \cdot \boldsymbol{u} dV \tag{1.1}$$

measures the entanglement of the vorticity lines of the flow quantitatively [1], where the boundary of the domain of integration is a closed vorticity surface or a periodical boundary. Similarly, one can define stream helicity [2] for a solenoidal velocity field $\boldsymbol{u}$ as

$$H_s \equiv \int \boldsymbol{u} \cdot \boldsymbol{A} dV, \tag{1.2}$$

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where $\boldsymbol{A}$ is the vector potential or the vector stream function of $\boldsymbol{u}$ satisfying $\boldsymbol{u}=\nabla\times\boldsymbol{A}$. The boundary of the domain of integration is a closed stream surface or a periodical boundary. The stream helicity measures the entanglement of the stream lines of an incompressible flow field. When $\boldsymbol{u}$ is taken as a magnetic field, the stream helicity is no other than the magnetic helicity measuring the entanglement of the magnetic field lines [3,4]. It can be noted that $\boldsymbol{A}$ is in general not an observable quantity and is not unique. Therefore, one question occurs: whether and when the H_s can be determined uniquely? The present study will not only give a clear answer to this question but also give a thorough calculation for the stream helicity in domains of general topology, including simply-connected and multiply-connected domains. For simplicity, we shall demonstrate the working procedure just for toroidal domains as an example. Actually, even for such a case this work has never been done thoroughly (see, e.g., [4–6]). It is well known that TOKAMAK as the most concerned apparatus for controlled-fusion is a toroidal container, wherein the configuration of magnetic field exerts crucial influences on the motion of the particles and hence on the efficiency of magnetic constrainment and nuclear fusion.

2 Analysis of stream helicity in a toroidal domain

We shall solve the problem of the determination of stream helicity in accordance of the connectivity property of a bounded domain in 3D space. The main tool is helical-wave decomposition (HWD). First we shall review the central theorem on the HWD. Then we point out some fundamental results and therewith complete the analysis.

2.1 Helical-wave decomposition for multiply-connected domains

The so-called helical wave is the vector eigenfunction of operator curl. For the subset of solenoidal fields in a bounded domain Ω in 3D space with no penetration on the boundary $\partial\Omega$ for both itself and its curl and with no flux passing through each insulation section Σ_i of Ω to turn a multiply connected domain into a simply connected one, the eigenvalue problem for operator curl is well-posed and the eigenvalues consist a discrete set $\{\cdots\leq\lambda_{-2}\leq\lambda_{-1}<0<\lambda_1\leq\lambda_2\leq\cdots\}$ with the corresponding eigenfunctions (helical waves) span a set of complete and orthogonal basis $\{\boldsymbol{B}_n,\ n\in\mathbb{Z}\}$ for any solenoidal field $\boldsymbol{u}$ with no penetration on the boundary and null flux through each Σ_i. Here, $i=0,1,2,\cdots,\nu$ and ν is the first Betti number (genus) of Ω, illuminating the connectivity of Ω. A simply connected Ω corresponds to $\nu=0$. The above statement outlines the theorem after Yoshida and Giga [7], by which any solenoidal field $\boldsymbol{u}$ satisfying $\boldsymbol{n}\cdot\boldsymbol{u}=0$ everywhere on $\partial\Omega$ can be expanded into a helical-wave series as

$$\boldsymbol{u}=\sum_{i=0}^{\nu}Q_i\nabla\chi_i+\sum_{n=-\infty}^{\infty}c_n\boldsymbol{B}_n, \tag{2.1}$$