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The Convergence and Superconvergence of a MFEM for Elliptic Optimal Control Problems

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Abstract. In this paper, we investigate a mixed finite element method (MFEM) for the elliptic optimal control problems (OCPs) with a distributive control. The state variable and adjoint state variable are approximated by the conforming rectangular $Q_{11}+Q_{01}\times Q_{10}$ elements pair. The discrete B-B condition is satisfied automatically, which is usually considered to be the key point of the MFEM. The control is then obtained by the orthogonal projection through the adjoint state. Optimal orders of convergence are derived for the above mentioned variables. Furthermore, superclose and superconvergence results are also established under certain reasonable regularity assumptions. Some numerical results are provided to verity the theoretical analysis. At last, the proposed method is extended to some other low order conforming and nonconforming elements.

AMS subject classifications: 65N30, 65N15

Key words: MFEMs, OCPs, optimal order error estimates, supercloseness and superconvergence.

1 Introduction

Consider the following constrained OCPs: find $(p,y,u)\in\mathbf{H}\times M\times U_{ad}$, such that

$$\min_{u\in U_{ad}\subset U}\left\{\frac{1}{2}\int_\Omega|\mathbf{p}-\mathbf{p_d}|^2d\mathbf{x}+\frac{1}{2}\int_\Omega(y-y_d)^2d\mathbf{x}+\frac{\alpha}{2}\int_\Omega u^2d\mathbf{x}\right\} \tag{1.1}$$

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subject to

$$\begin{cases} -\text{div}\,\mathbf{p}=f+u & \text{in } \Omega, \\ \mathbf{p}=\nabla y & \text{in } \Omega, \\ y=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a rectangular domain with a boundary $\partial\Omega$ in R^2. $\mathbf{p_d}\in(C^0(\overline{\Omega}))^2$, $y_d\in C^0(\overline{\Omega})$ and $f\in L^2(\Omega)$ are given functions. We denote $\mathbf{H}=(L^2(\Omega))^2$, $M=H_0^1(\Omega)$, $U=L^2(\Omega)$. Throughout this paper, we adapt standard Sobolev spaces and norms defined in [1]. The set U_{ad} is defined as

$$U_{ad}=\{v\in U: a(\mathbf{x})\le v\le b(\mathbf{x}) \ \text{ a.e. in } \Omega\}, \tag{1.3}$$

where $a(\mathbf{x})$, $b(\mathbf{x})\in L^\infty(\Omega)$, and $a(\mathbf{x})<b(\mathbf{x})$ for a.e. $\mathbf{x}=(x_1,x_2)\in\Omega$.

Partial differential equation (PDE) constrained control problems have been playing a crucial role in many science and engineering applications (cf. [2,3]). But the exact solutions do not always exist or are difficult to be found, so some studies have been devoted to the numerical simulations. For example, [4] firstly proposed the FEM for PDE control problems and derived some error estimates. Later, [5] obtained the $\mathcal{O}(h^2)$ order superconvergence by employing linear triangular elements and a projection of the discrete adjoint state for the elliptic OCPs. In recent several decades, the so-named optimality conditions (c.f. [6]) led to some a prior and a posterior error estimates of FEMs for OCPs governed by different PDEs (see [7-11]).

As we know, in some control problems, the objective functional contains the gradient of the state variable, so the precision of the numerical gradient is also very important in numerical approximation of the state equation. In fact, MFEMs approximate both the scalar variable and its gradient with the same accuracy, and are therefore more suitable for these problems than conventional FEMs. For MFEM, [12] treated the basic ideas at an introductory level, and discussed the advantages and disadvantages of the mixed methods. [13] constructed the famous R-T mixed spaces and obtained optimal error estimates for second order elliptic equations. Many authors have developed the applications and other properties of mixed finite element approximations in certain areas, which refer to [14-18] and the references cited therein. By employing the R-T element pair to approximate the state and adjoint, [19] researched a MFEM for the convex OCPs governed by elliptic equations, in which the convergence and superconvergence of FEM approximations were derived. Furthermore, superconvergence of the R-T rectangular and triangular MFEMs for the quadratic OCPs were investigated in [20] and [21]. [22] derived some a priori and a posteriori error estimates of R-T H^1-Galerkin MFEM for elliptic OCPs. [23] proposed a stabilized MFEM for elliptic control problems by adding suitable elementwise least-squares residual terms for the primal state variable and its flux. [24] considered the MFEMs for Dirichlet boundary OCPs, and obtained optimal and quasi-optimal error estimates for problems on polygonal and smooth domains, respectively. [25] and [26] studied the nonconforming MFEMs for elliptic and Stokes OCPs, respectively.