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CONVERGENCE AND OPTIMALITY OF ADAPTIVE MIXED METHODS FOR POISSON'S EQUATION IN THE FEEC FRAMEWORK*

Michael Holst, Yuwen Li and Adam Mihalik
Department of Mathematics, University of California, San Diego, CA 92123, USA
Email: mholst@math.ucsd.edu, yul739@ucsd.edu, amihalik@math.ucsd.edu
Ryan Szypowski
Department of Mathematics and Statistics, Cal Poly Pomona, USA
Email: rsszypowski@csupomona.edu

Abstract

Finite Element Exterior Calculus (FEEC) was developed by Arnold, Falk, Winther and others over the last decade to exploit the observation that mixed variational problems can be posed on a Hilbert complex, and Galerkin-type mixed methods can then be obtained by solving finite-dimensional subcomplex problems. Chen, Holst, and Xu (Math. Comp. 78 (2009) 35–53) established convergence and optimality of an adaptive mixed finite element method using Raviart–Thomas or Brezzi–Douglas–Marini elements for Poisson's equation on contractible domains in $\mathbb{R}^2$, which can be viewed as a boundary problem on the de Rham complex. Recently Demlow and Hirani (Found. Math. Comput. 14 (2014) 1337–1371) developed fundamental tools for a posteriori analysis on the de Rham complex. In this paper, we use tools in FEEC to construct convergence and complexity results on domains with general topology and spatial dimension. In particular, we construct a reliable and efficient error estimator and a sharper quasi-orthogonality result using a novel technique. Without marking for data oscillation, our adaptive method is a contraction with respect to a total error incorporating the error estimator and data oscillation.

Mathematics subject classification: 65N12, 65N15, 65N30, 65N50.
Key words: Finite element exterior calculus, Adaptive finite element methods, A posteriori error estimates, Convergence, Quasi-optimality.

1. Introduction

An idea that has had a major influence on the development of numerical methods for PDE applications is that of mixed finite elements, whose early success in areas such as computational electromagnetics was later found to have surprising connections with the calculus of exterior differential forms, including de Rham cohomology and Hodge theory [9, 19, 30, 31]. A core idea underlying these developments is the Helmholtz-Hodge orthogonal decomposition of an arbitrary vector field $f \in (L^2(\Omega))^3$ into curl-free, divergence-free, and harmonic functions:

$$f = \nabla p + \nabla \times q + h,$$

where $p \in H_0^1(\Omega), q \in H(\mathrm{curl}, \Omega)$, and h is harmonic (divergence- and curl-free). The mixed formulation is explicitly computing the decomposition for $h = 0$, and finite element methods based on mixed formulations exploit this. There is a connection between this decomposition

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and de Rham cohomology; the space of harmonic forms is isomorphic to the first de Rham cohomology of the domain Ω, with the number of holes in Ω giving the first Betti number, and creating obstacles to well-posed formulations of elliptic problems. A natural question is then: What is an appropriate mathematical framework for understanding this abstractly, that will allow for a methodical construction of "good" finite element methods for these types of problems? The answer turns out to be the theory of Hilbert Complexes. Hilbert complexes were originally studied in [11] as a way to generalize certain properties of elliptic complexes, particularly the Hodge decomposition and other aspects of Hodge theory. The Finite Element Exterior Calculus (FEEC) [3, 4] was developed to exploit this abstraction. A key insight was that from a functional-analytic point of view, a mixed variational problem can be posed on a Hilbert complex: a differential complex of Hilbert spaces, in the sense of [11]. Galerkin-type mixed methods are then obtained by solving the variational problem on a finite-dimensional subcomplex. Stability and consistency of the resulting method, often shown using complex and case-specific arguments, are reduced by the framework to simply establishing existence of operators with certain properties that connect the Hilbert complex with its subcomplex, essentially giving a "recipe" for the development of provably well-behaved methods.

Due to the pioneering work of Babuška and Rheinboldt [5], adaptive finite element methods (AFEM) based on a posteriori error estimators have become standard tools in solving PDE problems arising in science and engineering (cf. [1, 34, 38]). A standard adaptive algorithm has the general iterative structure:

$$\mathsf{Solve} \longrightarrow \mathsf{Estimate} \longrightarrow \mathsf{Mark} \longrightarrow \mathsf{Refine}, \tag{1.1}$$

where Solve computes the discrete solution u_ℓ in a subspace $X_\ell \subset X$; Estimate computes certain error estimators based on u_ℓ, which are reliable and efficient in the sense that they are good approximation of the true error $u - u_\ell$ in the energy norm; Mark applies certain marking strategies based on the estimators; and finally, Refine divides each marked element and completes the mesh to obtain a new partition, and subsequently an enriched subspace $X_{\ell+1}$. The fundamental problem with the adaptive procedure (1.1) is guaranteeing convergence of the solution sequence. The first convergence result for (1.1) was obtained by Babuška and Vogelius [6] for linear elliptic problems in one space dimension. The multi-dimensional case was open until Dörfler [18] proved convergence of (1.1) for Poisson's equation by using the so called Dörfler marking, under the assumption that the initial mesh was fine enough to resolve the influence of data oscillation. This result was improved by Morin, Nochetto, and Siebert [28], in which the convergence was proved without conditions on the initial mesh, but requiring the so-called interior node property, together with an additional marking step driven by data oscillation. It was shown by Binev, Dahmen and DeVore [8] for the first time that AFEM for Poisson's equation in the plane has optimal computational complexity by using a special coarsening step. This result was improved by Stevenson [36] by showing the optimal complexity in general spatial dimension without a coarsening step. These error reduction and optimal complexity results were improved in several aspects in [12]. In their analysis, the artificial assumptions of interior node and extra marking due to data oscillation were removed, and the convergence result is applicable to general linear elliptic equations. The main ingredients of this new convergence analysis are the global upper bound on the error given by the a posteriori estimator, orthogonality (or possibly only quasi-orthogonality) of the underlying bilinear form arising from the linear problem, and a type of error indicator reduction produced by each step of AFEM. In another direction, Morin, Siebert, and Veeser [29] gave a plain convergence result of conforming AFEMs for a widge range of linear problems without using Dörfler makring. We refer to [32] for a recent survey of convergence analysis of AFEM for linear elliptic PDE problems which gives an overview of all of these results through 2012. See also [23] or an overview of various extensions to nonlinear problems.

Of particular relevance here is the 2009 article of Chen, Holst, and Xu [13], where convergence and optimality of an adaptive mixed finite element method (AMFEM) using Raviart–Thomas (RT) [33] or Brezzi–Douglas–Marini (BDM) [10] elements for Poisson's equation on simply connected polygons in $\mathbb{R}^2$ was established. The main difficulty for convergence analysis of AMFEM is the lack of minimization principle, and thus the failure of orthogonality. A main contribution of [13] is a quasi-orthogonality result on the error $\|\sigma - \sigma_h\|$. The proof is based on the fact that the error is orthogonal to the divergence free subspace, while the part of the error that is not divergence free was bounded by the data oscillation using a discrete stability result. We also mention that Becker and Mao [7] developed a convergent AMFEM with optimal comlexity using the lowest-order RT finite element in $\mathbb{R}^2$. They used a multigrid inexact solver as the SOLVE module, which is another direction of interest. Recently, [25] extended the analysis of adaptive mixed methods for Poisson's equation in [13] to $\mathbb{R}^3$ and [24] provided a unified analysis of many adaptive mixed methods in $\mathbb{R}^2$ and $\mathbb{R}^3$. [7,13,24,25] all used discrete Helmholtz decomposition on contractible domains that ignores harmonic components (harmonic forms in FEEC).

In this paper, we generalize the results in [2,13] by analyzing the error $\|\sigma-\sigma_h\|$ in the FEEC framework, which allows us to extend the a posteriori upper bound in [2] and convergence and complexity results in [13] on contractible domains in $\mathbb{R}^2$ to domains of arbitrary topology and spatial dimension. By saying "FEEC framework", we identify the classical mixed formulation and mixed method for Poisson's equation as (2.8) posed on the de Rham complex and (2.9) on the discrete de Rham complex, respectively. In this way, tools for dealing with harmonic forms in [3,4] and deriving a posteriori error indicators in [17] can be applied . In FEEC terminology, the method considered in [13] are equivalent to those for solving the Hodge Laplacian problem when $k = n = 2$. All of our results apply to the case $k = n$ for arbitrary $n \geq 2$ and domains which are not necessarily contractible. Even in the case $k = n = 2$, our quasi-orthogonality result is sharper than [13] in the sense that it involves a local data oscillation. The quasi-orthogonality Theorem 4.1 is motivated by Becker and Mao's result [7] in $\mathbb{R}^2$. The key ingredient of the proof is Lemma 4.1, a discrete approximation result in the standard L^2-norm, while [24, 25] used a carefully designed mesh-dependent norm and a discrete inf-sup condition to achieve their quasi-orthogonality in $\mathbb{R}^3$. With the sharper quasi-orthogonality, we are able to prove contraction of Algorithm AMFEM by defining the total error $\|\sigma - \sigma_h\|^2 + \rho\eta^2_{\mathcal{T}_h}(\sigma_h, \mathcal{T}_h) + \zeta\,\mathrm{osc}^2_{\mathcal{T}_h}(f, \mathcal{T}_h)$, see Theorem 5.2. Comparing to [13] using a separate marking driven by data oscillation, AMFEM uses a single marking step based on the estimator $\eta_{\mathcal{T}_h}(\sigma_h, T)$.

While it is possible to restate the improvement in this paper in the classical context of mixed methods as [7,13,24,25], the framework of FEEC has its own advantages. By using the language of differential forms, our analysis is unified and straightforward, e.g., topology-independent and dimension-independent. As mentioned above, classical a posteriori error analysis of mixed methods implicitly assumed that Ω has no "holes" in $\mathbb{R}^2$ and $\mathbb{R}^3$ and ignored harmonic part in the Helmholtz decomposition, while the harmonic form encoding the topological information is built in the Hodge decomposition and handled by the gap theorem 3.1 in FEEC. In addition, residual-based error indicators of AMFEM make use of the tangential trace and adjoint of curl (cf. [2, 25]), whose definitions and behaviors are quite different in $\mathbb{R}^2$ and $\mathbb{R}^3$ and unclear in $\mathbb{R}^n$

with $n \geq 4$. In FEEC, the tangential trace is unified as $\operatorname{tr}\star$ and the adjoint of curl is unified as the coderivative δ in any dimension, see the error indicator (3.1).

This paper is a revised version of the unpublished preprint [20] in 2013 whose goal is to shed some light on a posteriori error analysis, convergence and optimality of adaptive methods on the de Rham complex. In the revised version, we give a completely new proof of Theorem 4.1, a refined quasi-orthogonality result, while [20] follows the quasi-orthogonality proof in [13]. Second, the contraction analysis of AMFEM is novel by using the aforementioned improved quasi-orthogonality and total error. In addition, several inaccuracies in [20] such as proofs of Corollary 3.1 and quasi-optimality are fixed or removed.

Recently, there are several results on convergence and optimality of AMFEM in FEEC. Demlow [16] developed a convergent AFEM with optimal complexity for computing the space of harmonic forms. In [14], Chen and Wu developed a convergent AMFEM for solving the Hodge Laplacian with index $1 \leq k \leq n-1$ with respect to the error $\|d(\sigma - \sigma_h)\|^2 + \|d(u - u_h)\|^2$ on contractible domains. The second author [27] developed two AMFEMs for the Hodge Laplacian with index $1 \leq k \leq n$ on Lipschitz domains with general topology. When $k = n$, his results can control and reduce the energy error $\|\sigma - \sigma_h\|_{H\Lambda^{n-1}}$ while AMFEM is dealing with the L^2 error $\|\sigma - \sigma_h\|$. Assuming sufficient regularity, $\|\sigma - \sigma_h\| = O(h^{r+2})$ is of higher order than $\|\sigma - \sigma_h\|_{H\Lambda^{n-1}} = O(h^{r+1})$ when using the generalized BDM pair (2.11). In addition, the quasi-orthogonality result Theorem 4.1 is sharper than [27] and the proof is quite different.

The remainder of the paper is organized as follows. In Section 2 we introduce the notational and technical tools in FEEC needed for the paper. In Section 3 we present an error indicator with global reliability and local efficiency. In Section 4, we construct the quasi-orthogonality result. The adaptive algorithm AMFEM is then presented in Section 5, and we prove both convergence and optimality.

2. Preliminaries

In this section we first review abstract Hilbert complexes. We then examine the particular case of the de Rham complex. We follow closely the notation and the general development of Arnold, Falk, and Winther in [3, 4]. We also discuss results from Demlow and Hirani in [17]. (See also [21, 22] for a concise summary of Hilbert Complexes in a yet more general setting.) We then give an overview of the basics of Adaptive Finite Element Methods (AFEM), and the ingredients we will need to prove convergence and optimality within the FEEC framework.

2.1. Hilbert complexes

We begin with a quick summary of some basic concepts and definitions. A Hilbert complex (W, d) is a sequence of Hilbert spaces W^k equipped with the inner product $\langle \cdot, \cdot \rangle$, closed and densely defined linear operators, d^k, which map their domain, $V^k \subset W^k$ to the kernel of d^{k+1} in W^{k+1}. A Hilbert complex is bounded if each d^k is a bounded linear map from W^k to W^{k+1} A Hilbert complex is closed if the range of each d^k is closed in W^{k+1}. Given a Hilbert complex (W, d), the subspaces $V^k \subset W^k$ endowed with the graph inner product

$$\langle u, v \rangle_V = \langle u, v \rangle + \langle d^k u, d^k v \rangle,$$

form a Hilbert complex (V, d) known as the domain complex. By definition $d^{k+1} \circ d^k = 0$, thus (V, d) is a bounded Hilbert complex. Additionally, (V, d) is closed if (W, d) is closed.

The range of d^{k-1} in V^k will be represented by $\mathfrak{B}^k$, and the null space of d^k will be represented by $\mathfrak{Z}^k$. Clearly, $\mathfrak{B}^k \subset \mathfrak{Z}^k$. The elements of $\mathfrak{Z}^k$ orthogonal to $\mathfrak{B}^k$ are the space of harmonic forms, represented by $\mathfrak{H}^k$. For a closed Hilbert complex we can write the Hodge decomposition of W^k and V^k,

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp}, \tag{2.1}$$

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V}, \tag{2.2}$$

where $\perp$ denotes the orthogonal complement w.r.t. $\langle \cdot, \cdot \rangle$ and $\mathfrak{Z}^{k\perp V} := \mathfrak{Z}^{k\perp} \cap V^k$. We use $P_{\mathfrak{B}}, P_{\mathfrak{H}}, P_{\mathfrak{Z}^\perp}$ to denote the L^2 projections onto $\mathfrak{B}^k, \mathfrak{H}^k, \mathfrak{Z}^{k\perp}$,respectively. Another important Hilbert complex will be the dual complex (W, d^*), where $d_k^* : W^k \to W^{k-1}$, is the adjoint of d^{k-1}. The domain of d_k^* will be denoted by V_k^*. Let $\mathfrak{Z}_k^*$ denote the null space of d_k^* and $\mathfrak{B}_k^*$ the range of d_{k+1}^*. For closed Hilbert complexes, an important result will be the Poincaré inequality,

$$\|v\|_V \leq c_P \|d^k v\|_W, \qquad v \in \mathfrak{Z}^{k\perp}. \tag{2.3}$$

In addition, we have the important relation $\mathfrak{Z}_k^\perp = \mathfrak{B}_k^*$. The de Rham complex is the practical complex where general results we show on an abstract Hilbert complex will be applied.

The abstract Hodge Laplacian

Given a Hilbert complex (W, d), the operator $L = dd^* + d^*d$, $W^k \to W^k$ will be referred to as the abstract Hodge Laplacian. For $f \in W^k$, the Hodge Laplacian problem can be formulated as the problem of finding $u \in W^k$ such that

$$\langle du, dv \rangle + \langle d^*u, d^*v \rangle = \langle f, v \rangle, \qquad v \in V^k \cap V_k^*.$$

A necessary condition for the solution to exsit is $f \perp \mathfrak{H}^k$. The above formulation has undesirable properties from a computational perspective. The finite element spaces $V^k \cap V_k^*$ is difficult to construct, and the problem will not be well-posed in the presence of a non-trivial harmonic space $\mathfrak{H}^k$. In order to circumvent these issues, a well-posed (cf. [3, 4]) mixed formulation of the abstract Hodge Laplacian is introduced as the problem of finding $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$, such that:

$$\begin{aligned} &\langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0, && \forall \tau \in V^{k-1}, \\ &\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle = \langle f, v \rangle, && \forall v \in V^k, \\ &\langle u, q \rangle = 0, && \forall q \in \mathfrak{H}^k. \end{aligned} \tag{2.4}$$

Subcomplexes and approximate solutions to the Hodge Laplacian

In [3,4] a theory of approximate solutions to the Hodge Laplacian problem is developed by using finite dimensional approximation of Hilbert complexes. Let (W, d) be a Hilbert complex with domain complex (V, d). An approximating subcomplex is a set of finite dimensional Hilbert spaces, $V_h^k \subset V^k$ with the property that $d^k V_h^k \subset V_h^{k+1}$. We identify W_h^k with V_h^k but endowed with the norm $\langle \cdot, \cdot \rangle$. Following [4], we use $\mathfrak{Z}_h, \mathfrak{B}_h, \mathfrak{H}_h, \mathfrak{B}_h^*$ with obvious meaning. Since (V_h, d) is a closed Hilbert complex, V_h^k has a corresponding Hodge decomposition:

$$V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp}.$$

Using elementary linear algebra, we have $\mathfrak{Z}_h^{\perp} = \mathfrak{B}_h^*$. By this construction, (V_h, d) is an abstract Hilbert complex with a well-posed Hodge Laplacian problem: Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$, such that

$$\begin{aligned} &\langle \sigma_h, \tau\rangle - \langle d\tau, u_h\rangle = 0, && \forall \tau \in V_h^{k-1}, \\ &\langle d\sigma_h, v\rangle + \langle du_h, dv\rangle + \langle p_h, v\rangle = \langle f, v\rangle, && \forall v \in V_h^k, \\ &\langle u_h, q\rangle = 0, && \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{2.5}$$

An assumption made in [4] in developing this theory is the existence of a bounded cochain projection $\pi_h : V \to V_h$, which commutes with the differential operator d.

In [4], an a priori convergence result is developed for the solutions on the approximating complexes. The result relies on the approximating complex getting sufficiently close to the original complex in the sense that $\inf_{v \in V_h^k} \|u - v\|_V$ can be assumed sufficiently small for relevant $u \in V^k$. Adaptive methods, on the other hand, gain computational efficiency by limiting the degrees of freedom used in areas of the domain where it does not significantly impact the quality of the numerical solution.

2.2. The de Rham complex and approximation properties

The de Rham complex is a cochain complex where the abstract results from the previous section can be applied in developing practical computational methods. This section reviews concepts and definitions related to the de Rham complex that will be needed in our development of an adaptive finite element method. This introduction will be brief and mostly follows the notation from the more in-depth discussion in [4].

For the remainder of the paper we assume a bounded Lipschitz polyhedral domain, $\Omega \in \mathbb{R}^n, n \geq 2$. Let $\Lambda^k(\Omega)$ be the space of smooth k-forms on Ω, and $L^2\Lambda^k(\Omega)$ be the completion of $\Lambda^k(\Omega)$ with respect to the L^2 inner-product. For $k = n$, the space of harmonic forms in $L^2\Lambda^n(\Omega)$ has no nonzero element, i.e. $\mathfrak{H}^n = \{0\}$, which simplifies the analysis in our case of interest $k = n$. However, $\sigma - \sigma_h$ is contained in $H\Lambda^{k-1}(\Omega)$, which generally contains a nontrivial harmonic component. Note that the convergence and optimality results in [13] hold only for simply connected polygons in $\mathbb{R}^2$, therefore $\mathfrak{H}^{n-1} = \{0\}$ is also true in the case $k = n = 2$.

The de Rham complex

Let d be the exterior derivative acting as an operator from $L^2\Lambda^k(\Omega)$ to $L^2\Lambda^{k+1}(\Omega)$. We still use $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_V$ to denote the L^2- and V-inner products respectively on the de Rham complex. This forms a Hilbert complex $(L^2\Lambda(\Omega), d)$, with domain complex $(H\Lambda(\Omega), d)$, where $H\Lambda^k(\Omega)$ is the set of elements in $L^2\Lambda^k(\Omega)$ with exterior derivatives in $L^2\Lambda^{k+1}(\Omega)$. The domain complex can be described with the following diagram

$$H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \to H\Lambda^{n-1}(\Omega) \xrightarrow{d} H\Lambda^n(\Omega) = L^2\Lambda^n(\Omega). \tag{2.6}$$

It can be shown that the compactness property is satisfied, and therefore the prior results shown on abstract Hilbert complexes can be applied.

The importance of the adjoint operator is clear by the first equation of the mixed Hodge Laplacian problem. For $\omega \in \Lambda^k(\Omega)$, the coderivative $\delta\omega \in \Lambda^{k-1}(\Omega)$ is the unique form satisfying $\star\delta\omega := (-1)^k d \star \omega$. Let

$$\begin{aligned} \mathring{H}\Lambda^k(\Omega) &:= \Big\{\omega \in H\Lambda^k(\Omega) : \ \mathrm{tr}\ \omega = 0 \text{ on } \partial\Omega\Big\}, \\ \mathring{H}^*\Lambda^k(\Omega) &:= \star\mathring{H}\Lambda^{n-k}(\Omega), \end{aligned}$$

where tr denotes the trace opeartor for differential forms. Stokes' theorem gives a useful version of integration by parts

$$\langle d\omega, \mu\rangle = \langle \omega, \delta\mu\rangle + \int_{\partial\Omega} \operatorname{tr}\omega \wedge \operatorname{tr}\star\mu, \quad \omega \in \Lambda^{k-1}(\Omega), \qquad \mu \in \Lambda^k(\Omega). \tag{2.7}$$

The following result uses the above concepts and is helpful in understanding the mixed Hodge Laplace problem on the de Rham complex.

Theorem 2.1 (Theorem 4.1 from [4]). *Let d be the exterior derivative viewed as an unbounded operator from $L^2\Lambda^{k-1}(\Omega)$ to $L^2\Lambda^k(\Omega)$ with domain $H\Lambda^k(\Omega)$. Then the adjoint d^*, as an unbounded operator from $L^2\Lambda^k(\Omega)$ to $L^2\Lambda^{k-1}(\Omega)$, has $\mathring{H}^*\Lambda^k(\Omega)$ as its domain and coincides with the operator δ.*

Using (2.4) with $k = n$, $du = 0$, $\mathfrak{H}^n = \{0\}$, and Theorem 2.1, we obtain the mixed formulation for the Hodge–Laplace equation on the de Rham complex: find $(\sigma, u) \in H\Lambda^{n-1}(\Omega) \times H\Lambda^n(\Omega)$ such that

$$\begin{aligned} &\langle \sigma, \tau\rangle - \langle d\tau, u\rangle = 0, && \forall \tau \in H\Lambda^{n-1}(\Omega), \\ &\langle d\sigma, v\rangle = \langle f, v\rangle, && \forall v \in H\Lambda^n(\Omega). \end{aligned} \tag{2.8}$$

By the isomorphisms $H\Lambda^{n-1}(\Omega) \cong H(\operatorname{div};\Omega), H\Lambda^n(\Omega) \cong L^2(\Omega)$ (cf. [3]), d is identified as the divergence operator and (2.8) can be realized as the classical mixed formulation for Poisson's equation with homogeneous Dirichlet boundary condition.

We use $(V(\mathcal{T}_h), d)$ [corresponds to (V_h, d)] to denote a finite dimensional subcomplex of $(H\Lambda, d)$ on the mesh $\mathcal{T}_h$. Assuming the exsitence of a cochain projection π_h, $\mathfrak{H}_h^n = \mathfrak{H}^n = \{0\}$. The discrete problem (2.5) with $k = n$ then reduces to find $(\sigma_h, u_h) \in V^{n-1}(\mathcal{T}_h) \times V^n(\mathcal{T}_h)$, such that

$$\begin{aligned} &\langle \sigma_h, \tau\rangle - \langle d\tau, u_h\rangle = 0, && \tau \in V^{n-1}(\mathcal{T}_h), \\ &\langle d\sigma_h, v\rangle = \langle f, v\rangle, && v \in V^n(\mathcal{T}_h). \end{aligned} \tag{2.9}$$

Let δ_h be the adjoint of $d : V^{n-1}(\mathcal{T}_h) \to V^n(\mathcal{T}_h)$, and $f_{\mathcal{T}_h}$ be the L^2-projection of f onto $V^n(\mathcal{T}_h)$. (2.9) is equivalent to $\sigma_h = \delta_h v_h$, $d\sigma_h = f_{\mathcal{T}_h}$. Note that $\sigma \in \mathfrak{Z}_h^\perp$ and $\sigma \in \mathfrak{Z}^\perp$.

Finite element differential forms

Given a shape regular, conforming simplicial triangulation $\mathcal{T}_h$ of Ω, we set $h_T := |T|^{\frac{1}{n}}$ for an element $T \in \mathcal{T}_h$, where $|T|$ is the volume of T. The finite element space $V^k(\mathcal{T}_h) \subset H\Lambda^k(\Omega)$ is a space of k-forms with piecewise polynomial coefficients. In particular, we assume that

$$V^{n-1}(\mathcal{T}_h) = \mathcal{P}_{r+1}^-\Lambda^{n-1}(\mathcal{T}_h), \qquad V^n(\mathcal{T}_h) = \mathcal{P}_r\Lambda^n(\mathcal{T}_h), \tag{2.10}$$

or

$$V^{n-1}(\mathcal{T}_h) = \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}_h), \qquad V^n(\mathcal{T}_h) = \mathcal{P}_r\Lambda^n(\mathcal{T}_h), \tag{2.11}$$

for any nonnegative integer r. In fact, $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ consists of k-forms in $H\Lambda^k(\Omega)$ with piecewise polynomial coefficients of degree r and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ is in a special Koszul complex on $\mathcal{T}_h$. Pairs (2.10) and (2.11) are generalizations of RT and BDM elements respctively in FEEC. For a detailed discussion on these spaces, see [4].

Bounded Cochain Projections

Bounded cochain projections and their approximation properties are necessary in the analysis of both uniform and adaptive FEMs in the FEEC framework. We will use frequently the following two operators: the smoothed projection $\pi_h : L^2\Lambda^k(\Omega) \to V^k(\mathcal{T}_h)$ from [15], and the commuting quasi-interpolation $\Pi_h : L^2\Lambda^k(\Omega) \to V^k(\mathcal{T}_h)$ as defined in [17] with ideas similar to [35].

In the remainder of the paper, C will be a generic constant which is dependent only on Ω and the shape regularity of the underlying mesh . We use $\langle \cdot, \cdot \rangle_{\Omega_0}$ to denote the L^2 inner product restricted to Ω_0. $\| \cdot \|$ will denote the $L^2\Lambda^k(\Omega)$ norm, and when taken on specific elements of the domain T and ∂T, we write $\| \cdot \|_T$ and $\| \cdot \|_{\partial T}$ respectively. For all other norms, such as $H\Lambda^k(\Omega)$ and $H^1\Lambda^k(\Omega)$, we write $\| \cdot \|_{H\Lambda^k(\Omega)}$ and $\| \cdot \|_{H^1\Lambda^k(\Omega)}$ respectively.

The next lemma is taken directly from Lemma 6 in [17], and will be a key tool in developing an upper bound for the error.

Lemma 2.1. *Assume $1 \le k \le n$, and $\phi \in H\Lambda^{k-1}(\Omega)$ with $\|\phi\|_{H\Lambda^{k-1}(\Omega)} \le 1$. There exists $\varphi \in H^1\Lambda^{k-1}(\Omega)$ such that $d\varphi = d\phi, \Pi_h d\phi = d\Pi_h \phi = d\Pi_h \varphi$, and*

$$\sum_{T \in \mathcal{T}_h} h_T^{-2} \|\varphi - \Pi_h \varphi\|_T^2 + h_T^{-1} \|\mathrm{tr}(\varphi - \Pi_h \varphi)\|_{\partial T}^2 \le C.$$

3. Error Estimator

For $T \in \mathcal{T}_h$, let $[\![\tau]\!]$ denote the jump of τ over an element face. For element faces on $\partial\Omega$ we set $[\![\tau]\!] = \tau$. The element error indicator is defined as

$$\eta_{\mathcal{T}_h}^2(\sigma_h, T) = h_T \|[\![\mathrm{tr} \star \sigma_h]\!]\|_{\partial T}^2 + h_T^2 \|\delta \sigma_h\|_T^2 + h_T^2 \|f - f_{\mathcal{T}_h}\|_T^2. \tag{3.1}$$

For a subset $\mathcal{M} \subseteq \mathcal{T}_h$, define

$$\begin{aligned} \eta_{\mathcal{T}_h}^2(\sigma_h, \mathcal{M}) &:= \sum_{T \in \mathcal{M}} \eta_{\mathcal{T}_h}^2(\sigma_h, T), \\ \mathrm{osc}_{\mathcal{T}_h}^2(f, \mathcal{M}) &:= \sum_{T \in \mathcal{M}} h_T^2 \|f - f_{\mathcal{T}_h}\|_T^2. \end{aligned}$$

The Hodge decomposition is crucial to proving global reliability of $\eta_{\mathcal{T}_h}$. Using $\sigma \in \mathfrak{Z}^\perp$ and $\sigma_h \in \mathfrak{Z}_h^\perp$, the Hodge decomposition of $\sigma - \sigma_h$ can be written as

$$\begin{aligned} \sigma - \sigma_h &= P_{\mathfrak{B}}(\sigma - \sigma_h) + P_{\mathfrak{H}}(\sigma - \sigma_h) + P_{\mathfrak{Z}^\perp}(\sigma - \sigma_h) \\ &= (\sigma - P_{\mathfrak{Z}^\perp}\sigma_h) - P_{\mathfrak{B}}\sigma_h - P_{\mathfrak{H}}\sigma_h. \end{aligned} \tag{3.2}$$

Lemmas 3.1, 3.2 and 3.3 will bound each portion of this orthogonal decomposition.

Lemma 3.1.

$$\|\sigma - P_{\mathfrak{Z}^\perp}\sigma_h\| \le C \,\mathrm{osc}_{\mathcal{T}_h}(f, \mathcal{T}_h).$$

Proof. Since $\sigma - P_{\mathfrak{Z}^\perp}\sigma_h \in \mathfrak{Z}^{n-1\perp} = \mathfrak{B}_{n-1}^*$, $\sigma - P_{\mathfrak{Z}^\perp}\sigma_h = \delta v$ for some $v \in \mathrm{Dom}(\delta)$, where $\mathrm{Dom}(\delta) = \mathring{H}^*\Lambda^n(\Omega) \cong H_0^1(\Omega)$ is the domain of δ by Theorem 2.1. Thus

$$\|\sigma - P_{\mathfrak{Z}^\perp}\sigma_h\|^2 = \langle \sigma - P_{\mathfrak{Z}^\perp}\sigma_h, \delta v\rangle = \langle d\sigma - d\sigma_h, v\rangle.$$

Then by $\sum_{T\in\mathcal{T}_h} h_T^{-2}\|v - v_{\mathcal{T}_h}\|_T^2 \le C\|\delta v\|^2$, we obtain

$$\|\sigma - P_{\mathfrak{Z}^\perp}\sigma_h\|^2 = \langle f - f_{\mathcal{T}_h}, v - v_{\mathcal{T}_h}\rangle \le C\,\mathrm{osc}_{\mathcal{T}_h}(f,\mathcal{T}_h)\|\delta v\|.$$

The proof is complete. □

The next lemma uses the quasi-interpolant Π_h described in [17], and also applies integration by parts in the same standard fashion that [17] used when bounding error measured in the natural norm, $\|u-u_h\|_{H\Lambda^k(\Omega)} + \|\sigma-\sigma_h\|_{H\Lambda^{k-1}(\Omega)} + \|p - p_h\|$. In [17], inf-sup condition of the bilinear-form was used to separate components of the error, whereas here we simply analyze the orthogonal decomposition of $\sigma - \sigma_h$.

Lemma 3.2.

$$\|P_{\mathfrak{B}}\sigma_h\| \le C\eta_{\mathcal{T}_h}(\sigma_h,\mathcal{T}_h).$$

Proof. Note that

$$\|P_{\mathfrak{B}}\sigma_h\| = \langle \sigma_h, P_{\mathfrak{B}}\sigma_h/\|P_{\mathfrak{B}}\sigma_h\|\rangle = \langle -\sigma_h, d\phi\rangle, \quad \phi \in (\mathfrak{Z}^{k-2})^{\perp V}.$$

Since ϕ can then be replaced with φ satisfying the properties of Lemma 2.1, and noting $\sigma_h \perp \mathfrak{B}_h^{k-1}$,

$$\|P_{\mathfrak{B}}\sigma_h\| = \langle -\sigma_h, d(\varphi - \Pi_h\varphi)\rangle. \tag{3.3}$$

The problem is now reduced to a case handled in [17], when they bound a portion of their η_{-1} estimator. We follow their ideas to complete to proof. Applying the integration by parts formula we have

$$\|P_{\mathfrak{B}}\sigma_h\| = \sum_{T\in\mathcal{T}_h} -\int_{\partial T} \mathrm{tr}\star\sigma_h \wedge \mathrm{tr}(\varphi - \Pi_h\varphi) - \langle \delta\sigma_h, \varphi - \Pi_h\varphi\rangle_T.$$

Noting $\mathrm{tr}(\varphi - \Pi_h\varphi)$ is single-valued on the element boundaries and Cauchy–Schwarz inequality, this can be reduced to

$$\begin{aligned}
\|P_{\mathfrak{B}}\sigma_h\| &\le C\sum_{T\in\mathcal{T}_h} \|\,\mathrm{tr}(\varphi-\Pi_h\varphi)\|_{\partial T}\|[\![\mathrm{tr}\star\sigma_h]\!]\|_{\partial T} + \|\varphi - \Pi_h\varphi\|_T\|\delta\sigma_h\|_T \\
&\le C\sum_{T\in\mathcal{T}_h} \big(h_T^{\frac12}\|[\![\mathrm{tr}\star\sigma_h]\!]\|_{\partial T} + h_T\|\delta\sigma_h\|_T\big) \\
&\qquad \times \big(h_T^{-\frac12}\|\,\mathrm{tr}(\varphi-\Pi_h\varphi)\|_{\partial T} + h_T^{-1}\|\varphi-\Pi_h\varphi\|_T\big) \\
&\le C\eta_{\mathcal{T}_h}(\sigma_h,\mathcal{T}_h)\big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\,\mathrm{tr}(\varphi-\Pi_h\varphi)\|_{\partial T}^2 + h_T^{-2}\|\varphi-\Pi_h\varphi\|_T^2\big)^{1/2}.
\end{aligned}$$

The proof is then complete by applying the bounds from Lemma 2.1, and the Poincaré inequality $\|\phi\|_{H\Lambda^{k-1}} \le C\|d\phi\| = C$. □

To control the harmonic component in the Hodge decomposition, we need to estimate the gap between $\mathfrak{H}^{n-1}$ and $\mathfrak{H}_h^{n-1}$. To this end, we use equation (28) in [4]:

$$\|(I - P_{\mathfrak{H}^k})q\|_V \le \|(I - \pi_h^k)P_{\mathfrak{H}^k}q\|_V, \qquad q \in \mathfrak{H}_h^k. \tag{3.4}$$

Note that $\|\tilde q\| = \|\tilde q\|_V$ for any $\tilde q \in \mathfrak{H}^k$ or $\mathfrak{H}_h^k$. Combining (3.4) with a triangle inequality yields

$$\|q\| \le (\|(I-\pi_h^k)\| + 1)\|P_{\mathfrak{H}^k}q\| \le C\|P_{\mathfrak{H}^k}q\|, \qquad q \in \mathfrak{H}_h^k. \tag{3.5}$$

Theorem 3.1 will be essential in dealing with the harmonic forms in the proof of a continuous upper-bound. The corollary will be used identically when proving a discrete upper-bound. For use in our next two results we introduce the gap between subspaces and one of its important properties. Let A, B be $n < \infty$ dimensional, closed subspaces of a Hilbert space W. The gap between A and B is

$$\delta(A,B) = \sup_{x \in A,\ \|x\|=1} \|x - P_B x\|.$$

Then [17], Lemma 2 which takes the original ideas from [26], shows

$$\delta(A,B) = \delta(B,A). \tag{3.6}$$

Theorem 3.1. *For $1 \le k \le n-1$,*

$$\delta(\mathfrak{H}^k, \mathfrak{H}_h^k) = \delta(\mathfrak{H}_h^k, \mathfrak{H}^k) \le C < 1.$$

Proof. It is well known that $\dim(\mathfrak{H}_h^k) = \dim(\mathfrak{H}^k) = \beta_k$, the kth Betti number of the domain Ω. Then we can apply (3.6) to prove the equality. By (3.5) and the orthogonality of the L^2-projection, we have

$$\begin{aligned}\delta(\mathfrak{H}_h^k, \mathfrak{H}^k) &= \sup_{q \in \mathfrak{H}_h^k,\ \|q\|=1} \|q - P_{\mathfrak{H}} q\| \\ &= \sup_{q \in \mathfrak{H}_h^k,\ \|q\|=1} \sqrt{1 - \|P_{\mathfrak{H}} q\|^2} \\ &\le \sqrt{1 - \frac{1}{C^2}} < 1.\end{aligned}$$

The proof is complete. □

Corollary 3.1. *Let $\mathcal{T}_h$ be a conforming refinement of $\mathcal{T}_H$. Then*

$$\delta(\mathfrak{H}_h^k, \mathfrak{H}_H^k) = \delta(\mathfrak{H}_H^k, \mathfrak{H}_h^k) \le C < 1.$$

Proof. The proof follows the same logic as Theorem 3.1. The only difference is replacing (3.4) *by*

$$\|(I - P_{\mathfrak{H}_h})q\|_V \le \|(I - \pi_H^k) P_{\mathfrak{H}_h} q\|_V, \quad q \in \mathfrak{H}_H^k,$$

which can be derived by following the proof of (3.4). □

Lemma 3.3.

$$\|P_{\mathfrak{H}} \sigma_h\| \le C_{\mathfrak{H}} \|\sigma - \sigma_h\|, \quad C_{\mathfrak{H}} < 1.$$

Proof. Using $\sigma \perp \mathfrak{Z}^{k-1}$ and $\sigma_h \perp \mathfrak{Z}_h^{k-1}$, we have

$$\begin{aligned}\|P_{\mathfrak{H}} \sigma_h\| &= \sup_{q \in \mathfrak{H}, \|q\|=1} \langle \sigma_h, q - P_{\mathfrak{H}_h} q\rangle \\ &= \sup_{q \in \mathfrak{H}, \|q\|=1} \langle \sigma_h - \sigma, q - P_{\mathfrak{H}_h} q\rangle \\ &\le \delta(\mathfrak{H}^{n-1}, \mathfrak{H}_h^{n-1}) \|\sigma_h - \sigma\|.\end{aligned}$$

Then Lemma 3.3 follows from Theorem 3.1. □

Now we are in a position to prove the continuous upper bound.

Theorem 3.2 (continuous upper bound). *There exists a constant C_1, depending only on the shape regularity of $\mathcal{T}_h$, such that*

$$\|\sigma - \sigma_h\|^2 \le C_1 \eta^2_{\mathcal{T}_h}(\sigma_h, \mathcal{T}_h).$$

Proof. Starting from (3.2), by Lemmas 3.1, 3.2 and 3.3, we have

$$\begin{aligned}\|\sigma - \sigma_h\| &\le \|\sigma - P_{\mathfrak{Z}^\perp}\sigma_h\| + \|P_{\mathfrak{H}}\sigma_h\| + \|P_{\mathfrak{B}}\sigma_h\| \\ &\le \frac{1}{1 - C_{\mathfrak{H}}}(\|\sigma - P_{\mathfrak{Z}^\perp}\sigma_h\| + \|P_{\mathfrak{B}}\sigma_h\|) \\ &\le C_1 \eta_{\mathcal{T}_h}(\sigma_h, \mathcal{T}_h).\end{aligned}$$

The proof is complete. □

The efficiency can be proved by the standard bubble function technique in [17].

Theorem 3.3 (lower bound). *There exists a constant C_2 depending only on the shape regularity of $\mathcal{T}_h$, such that*

$$C_2 \eta^2_{\mathcal{T}_h}(\sigma_h, \mathcal{T}_h) \le \|\sigma - \sigma_h\|^2 + \operatorname{osc}^2_{\mathcal{T}_h}(f, \mathcal{T}_h).$$

4. Quasi-orthogonality

The main difficulty for proving convergence of AMFEM is the failure of orthogonality. In [13], a quasi-orthogonality property is proven using a technical discrete stability result. In this section, we use a novel technique to prove a sharper quasi-orthogonality result on $\langle \sigma - \sigma_h, \sigma_h - \sigma_H \rangle$. The next lemma provides a discrete approximation result.

Lemma 4.1. *Let $\mathcal{T}_h$ be a conforming refinement of $\mathcal{T}_H$ and P_H be the L^2 projection onto $\mathcal{P}_0\Lambda^n(\mathcal{T}_H)$. Then for any $T \in \mathcal{T}_H$ and $v_h \in V^n(\mathcal{T}_h)$,*

$$\|v_h - P_H v_h\|_T \le C h_T \|\delta_h v_h\|_T.$$

Proof. Let $\mathcal{T}_h|_T$ be the collection of simplices of $\mathcal{T}_h$ contained in T, namely, the restriction of $\mathcal{T}_h$ to T. Let $\mathring{V}_h^{n-1}(T) := \{\tau_h \in V^{n-1}(\mathcal{T}_h|_T) : \operatorname{tr}\tau_h = 0 \text{ on } \partial T\}$, and $V_h^n(T) = \mathcal{P}_r\Lambda^n(\mathcal{T}_h|_T)$ be two spaces of forms locally defined on T. By the theory of de Rham complexes with boundary condition ([3] and section 6.2 in [4]), $\mathring{H}\Lambda^{n-1}(T) \xrightarrow{d} L^2\Lambda^n(T) \xrightarrow{\int} \mathbb{R}$ and the discrete analogue $\mathring{V}_h^{n-1}(T) \xrightarrow{d} V_h^n(T) \xrightarrow{\int} \mathbb{R}$ are both exact. Recall that $\mathcal{P}_0\Lambda^n(\mathcal{T}_H)$ is the space of piecewise constant n-forms w.r.t. $\mathcal{T}_H$ and thus $\int_T(v_h - P_H v_h) = 0$. Hence using exactness, there exists $\tau_h \in \mathring{V}_h^{n-1}(T)$ with $d\tau_h = v_h - P_H v_h$. Clearly, we can pick $\tilde{\tau}_h \in \mathfrak{Z}_T^\perp$ such that $d\tilde{\tau}_h = d\tau_h$, where $\mathfrak{Z}_T^\perp$ is the orthogonal complement of the null space of d in $\mathring{V}_h^{n-1}(T)$. The Poincaré inequality (2.3) then gives

$$\|\tilde{\tau}_h\|_T \le C_T \|d\tilde{\tau}_h\|_T = C_T \|v_h - P_H v_h\|_T, \tag{4.1}$$

where the Poincare constant $C_T = O(h_T)$ by scaling. By zero extension, $\tilde{\tau}_h$ can be viewed as a global function in $V^{n-1}(\mathcal{T}_h)$. It then follows from (2.7), $(P_H v_h)|_T$ is constant, and $\operatorname{tr}\tilde{\tau}_h = 0$ on ∂T that

$$\langle P_H v_h, d\tilde{\tau}_h \rangle = \int_{\partial T} \operatorname{tr} \star (P_H v_h) \wedge \operatorname{tr}\tilde{\tau}_h = 0. \tag{4.2}$$

Finally using (4.2), $\mathrm{supp}\tilde{\tau}_h \subseteq T$, and (4.1), we have

$$\begin{aligned}\|v_h - P_H v_h\|_T^2 &= \langle v_h - P_H v_h, d\tilde{\tau}_h\rangle = \langle v_h, d\tilde{\tau}_h\rangle \\ &= \langle \delta_h v_h, \tilde{\tau}_h\rangle \le \|\delta_h v_h\|_T \|\tilde{\tau}_h\|_T \le C h_T \|\delta_h v_h\|_T \|v_h - P_H v_h\|_T,\end{aligned}$$

which completes the proof. □

The quasi-orthogonality result is a direct corollary of Lemma 4.1.

Theorem 4.1. *Let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$ and $\mathcal{R}_H$ be the set of refined elements in $\mathcal{T}_H$. For any $\varepsilon > 0$,*

$$(1-\varepsilon)\|\sigma - \sigma_h\|^2 \le \|\sigma - \sigma_H\|^2 - \|\sigma_h - \sigma_H\|^2 + \frac{C_0}{\varepsilon}\,\mathrm{osc}^2_{\mathcal{T}_H}(f, \mathcal{R}_H).$$

Proof. By $\sigma \perp \mathfrak{Z}_h$ and $\sigma_h \perp \mathfrak{Z}_h$, we have

$$\begin{aligned}|\langle \sigma - \sigma_h, \sigma_h - \sigma_H\rangle| &= |\langle \sigma - \sigma_h, P_{\mathfrak{Z}_h^\perp}(\sigma_h - \sigma_H)\rangle| \\ &\le \|\sigma - \sigma_h\| \|P_{\mathfrak{Z}_h^\perp}(\sigma_h - \sigma_H)\|.\end{aligned} \tag{4.3}$$

Since $P_{\mathfrak{Z}_h^\perp}(\sigma_h - \sigma_H) \in \mathfrak{Z}_h^\perp = \mathfrak{B}_h^*$, there exists $v_h \in V^n(\mathcal{T}_h)$ satisfying $P_{\mathfrak{Z}_h^\perp}(\sigma_h - \sigma_H) = \delta_h v_h$. Then using $d(\sigma_h - \sigma_H) = d\delta_h v_h$ and $\mathcal{P}_0\Lambda^n(\mathcal{T}_h) \subseteq V^n(\mathcal{T}_h)$, we have

$$\begin{aligned}\|P_{\mathfrak{Z}_h^\perp}(\sigma_h - \sigma_H)\|^2 &= \langle \delta_h v_h, \delta_h v_h\rangle \\ &= \langle d(\sigma_h - \sigma_H), v_h\rangle \\ &= \langle f_{\mathcal{T}_h} - f_{\mathcal{T}_H}, v_h - P_H v_h\rangle, \\ &= \langle f - f_{\mathcal{T}_H}, v_h - P_H v_h\rangle,\end{aligned} \tag{4.4}$$

where P_H is given in Lemma 4.1. For $T \in \mathcal{T}_H \backslash \mathcal{R}_H$, $v_h = P_H v_h$ on T. Hence combining (4.4), Cauchy–Schwarz inequality, and Lemma 4.1 yields

$$\begin{aligned}\|P_{\mathfrak{Z}_h^\perp}(\sigma_h - \sigma_H)\|^2 &= \sum_{T\in\mathcal{R}_H} \langle f - f_{\mathcal{T}_H}, v_h - P_H v_h\rangle_T \\ &\le \mathrm{osc}_{\mathcal{T}_H}(f, \mathcal{R}_H)\left(\sum_{T\in\mathcal{R}_H} h_T^{-2}\|v_h - P_H v_h\|_T^2\right)^{\frac{1}{2}} \\ &\le C_0^{\frac{1}{2}}\,\mathrm{osc}_{\mathcal{T}_H}(f, \mathcal{R}_H)\|\delta_h v_h\|.\end{aligned} \tag{4.5}$$

It then follows from (4.3) and (4.5) that

$$\begin{aligned}\|\sigma - \sigma_h\|^2 &= \|\sigma - \sigma_H\|^2 - \|\sigma_h - \sigma_H\|^2 - 2\langle \sigma - \sigma_h, \sigma_h - \sigma_H\rangle \\ &\le \|\sigma - \sigma_H\|^2 - \|\sigma_h - \sigma_H\|^2 + \varepsilon\|\sigma - \sigma_h\|^2 + \varepsilon^{-1}C_0\,\mathrm{osc}^2_{\mathcal{T}_H}(f, \mathcal{R}_H).\end{aligned}$$

The proof is complete. □

Comparing to the quasi-orthogonality

$$(1-\varepsilon)\|\sigma - \sigma_h\|^2 \le \|\sigma - \sigma_H\|^2 - \|\sigma_h - \sigma_H\|^2 + \frac{C}{\varepsilon}\,\mathrm{osc}^2_{\mathcal{T}_H}(f, \mathcal{T}_H) \tag{4.6}$$

proved in [13], Theorem 4.1 is sharper because $\mathrm{osc}_{\mathcal{T}_H}(f, \mathcal{R}_H) \le \mathrm{osc}_{\mathcal{T}_H}(f, \mathcal{T}_H)$. This improvement is crucial to the convergence analysis. Replacing $\mathrm{osc}_{\mathcal{T}_H}(f, \mathcal{T}_H)$ by $\mathrm{osc}_{\mathcal{T}_H}(f, \mathcal{R}_H)$ is motivated by the quasi-orthogonality result in [7] for the lowest order RT mixed method on simply connected polygon in $\mathbb{R}^2$. However, our technique is applicable to general domains in $\mathbb{R}^n$ and quite different from [7] as well as [13].

5. Convergence and Optimality

Given an initial triangulation, $\mathcal{T}_0$, the adaptive procedure will generate a nested sequence of triangulations $\mathcal{T}_\ell$ and discrete solutions σ_ℓ and u_ℓ, by looping through the following steps:

$$\mathsf{Solve} \longrightarrow \mathsf{Estimate} \longrightarrow \mathsf{Mark} \longrightarrow \mathsf{Refine}$$

Our adaptive mixed finite element method is as follows.

$[\mathcal{T}_N, \sigma_N] = \mathsf{AMFEM}(f, \mathcal{T}_0, \theta, \mathrm{tol})$

Given an initial mesh $\mathcal{T}_0$, a marking parameter $0 < \theta < 1$, an error tolerance tol > 0. Set $\ell = 0$, $\eta_\ell = \mathrm{tol} > 0$.

WHILE $\eta_\ell \geq \mathrm{tol}$, **DO**

1. Solve the discrete problem (2.9) on $\mathcal{T}_\ell$ to obtain the solution σ_ℓ.

2. For each $T \in \mathcal{T}_\ell$, compute $\eta_{\mathcal{T}_\ell}(\sigma_\ell, T)$ and $\eta_\ell = \eta_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{T}_\ell)$.

3. Select a subset $\mathcal{M}_\ell$ of $\mathcal{T}_\ell$ such that $\eta_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{M}_\ell) \geq \theta \eta_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{T}_\ell)$.

4. Refine $\mathcal{T}_\ell$ and necessary neighboring simplices by newest vertex bisection to get a conforming $\mathcal{T}_{\ell+1}$. Set $\ell \leftarrow \ell + 1$ and go to Step 1.

END DO

$\mathcal{T}_N = \mathcal{T}_\ell,\ \sigma_N = \sigma_\ell$.

Newest vertex bisection can maintain the shape regularity of $\{\mathcal{T}_\ell\}$, i.e., $\mathcal{T}_\ell$ is shape regular and the shape regularity depends only on $\mathcal{T}_0$. Bounding the number of simplexes generated in mesh refinements is important in the proof of quasi-optimality. Assuming that $\mathcal{T}_0$ satisfies a matching condition, Stevenson [37] has shown that newest vertex bisection guarantees

$$\#\mathcal{T}_\ell \leq \#\mathcal{T}_0 + C \sum_{i=0}^{\ell-1} \#\mathcal{M}_i. \tag{5.1}$$

5.1. Convergence of AMFEM

This subsection is devoted to convergence analysis of AMFEM. The results in this section follow ideas already in the literature [7, 12, 13], with Theorem 5.2 building on these ideas by proving reduction in a total error using relationships between data oscillation and reduction of a second type of total error. The following notation will be used in the proofs and discussion of this section:

$$e_\ell = \|\sigma - \sigma_\ell\|^2, \quad E_\ell = \|\sigma_{\ell+1} - \sigma_\ell\|^2, \quad \eta_\ell = \eta^2_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{T}_\ell),$$
$$o_\ell = \mathrm{osc}^2(f, \mathcal{T}_\ell), \quad \hat{o}_\ell = \mathrm{osc}^2(f, \mathcal{R}_\ell).$$

where $\mathcal{R}_\ell$ is the set of refined elements in $\mathcal{T}_\ell$.

Lemma 5.1.

$$\eta_{\ell+1} \leq \beta \eta_\ell + C_3 E_\ell, \tag{5.2}$$

where $0 < \beta < 1$ *and* $C_3 > 0$ *depend only on* θ *and* $\mathcal{T}_0$.

Proof. The proof is similar to Corollary 3.4 in [12]. Since η_ℓ involves data oscillation, we sketch the proof here for clarity. Let $\eta_\ell = \hat{\eta}_\ell + o_\ell$, where $\hat{\eta}_\ell = \hat{\eta}^2_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{T}_\ell)$ is the standard estimator without data oscillation. Given $T \in \mathcal{T}_{\ell+1}$, using a Young's inequality with parameter $\delta_* > 0$, we have

$$\hat{\eta}^2_{\mathcal{T}_{\ell+1}}(\sigma_{\ell+1}, T) \le (1+\delta_*)\hat{\eta}^2_{\mathcal{T}_{\ell+1}}(\sigma_\ell, T) + (1+\delta_*^{-1})C_{\mathcal{T}_0}\|\sigma_{\ell+1} - \sigma_\ell\|^2_T,$$

Summing over $T \in \mathcal{T}_{\ell+1}$ yields

$$\hat{\eta}^2_{\mathcal{T}_{\ell+1}}(\sigma_{\ell+1}, \mathcal{T}_{\ell+1}) \le (1+\delta_*)\hat{\eta}^2_{\mathcal{T}_{\ell+1}}(\sigma_\ell, \mathcal{T}_{\ell+1}) + (1+\delta_*^{-1})C_{\mathcal{T}_0}\|\sigma_{\ell+1} - \sigma_\ell\|^2.$$

It then follows from $o_{\ell+1} \le o_\ell$ that

$$\eta_{\ell+1} \le (1+\delta_*)\eta^2_{\mathcal{T}_{\ell+1}}(\sigma_\ell, \mathcal{T}_{\ell+1}) + (1+\delta_*^{-1})C_{\mathcal{T}_0}E_\ell. \tag{5.3}$$

For $T \in \mathcal{T}_\ell$, we use $\omega_T = \{t \in \mathcal{T}_{\ell+1} : t \subset T\}$. If $T \in \mathcal{M}_\ell$ is marked,

$$\sum_{t\in\omega_T} \hat{\eta}^2_{\mathcal{T}_{\ell+1}}(\sigma_\ell, t) \le 2^{-\frac{1}{n}}\hat{\eta}^2_{\mathcal{T}_\ell}(\sigma_\ell, T), \tag{5.4}$$

see Corollary 3.4 in [12]; and

$$\sum_{t\in\omega_T} \mathrm{osc}^2_{\mathcal{T}_{\ell+1}}(f, t) \le 2^{-\frac{2}{n}}\,\mathrm{osc}^2_{\mathcal{T}_\ell}(f, T), \tag{5.5}$$

see Lemma 5.2. If $T \in \mathcal{T}_\ell \backslash \mathcal{M}_\ell$, we use

$$\sum_{t\in\omega_T} \hat{\eta}_{\mathcal{T}_{\ell+1}}(\sigma_\ell, t) \le \hat{\eta}_{\mathcal{T}_\ell}(\sigma_\ell, T), \quad \sum_{t\in\omega_T} \mathrm{osc}_{\mathcal{T}_{\ell+1}}(f, t) \le \mathrm{osc}_{\mathcal{T}_\ell}(f, T).$$

Combining the above inequality with (5.4) and (5.5), we obtain

$$\begin{aligned} \eta^2_{\mathcal{T}_{\ell+1}}(\sigma_\ell, \mathcal{T}_{\ell+1}) &\le \eta^2_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{T}_\ell\backslash\mathcal{M}_\ell) + 2^{-\frac{1}{n}}\eta^2_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{M}_\ell) \\ &= \eta^2_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{T}_\ell) - \lambda\eta^2_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{M}_\ell), \end{aligned} \tag{5.6}$$

where $\lambda = 1 - 2^{-\frac{1}{n}} < 1$. It then follows from (5.3) and (5.6) that

$$\eta_{\ell+1} \le (1+\delta_*)\big(\eta_\ell - \lambda\eta^2_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{M}_\ell)\big) + (1+\delta_*^{-1})C_{\mathcal{T}_0}E_\ell. \tag{5.7}$$

Combining (5.7) and the marking condition $\eta^2_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{M}_\ell) \ge \theta^2\eta^2_\ell$, we obtain (5.2) with $\beta = (1+\delta_*)(1-\lambda\theta^2)$. $\beta < 1$ provided $\delta_* < \frac{\lambda\theta^2}{1-\lambda\theta^2}$. □

Now we are in a position to prove the error reduction.

Theorem 5.1. *When*

$$0 < \varepsilon < \frac{1-\beta}{C_1C_3},$$

there exist $\alpha \in$ (0,1) and $C_4, \rho > 0$ depending only on ε, θ and $\mathcal{T}_0$, such that

$$(1-\varepsilon)e_{\ell+1} + \rho\eta_{\ell+1} \le \alpha[(1-\varepsilon)e_\ell + \rho\eta_\ell] + C_4\hat{o}_\ell. \tag{5.8}$$

Proof. Recall the quasi-orthogonality Theorem 4.1 and global reliability Theorem 3.2,

$$e_\ell \le C_1\eta_\ell, \tag{5.9}$$
$$(1-\varepsilon)e_{\ell+1} \le e_\ell - E_\ell + C_0\varepsilon^{-1}\hat{o}_\ell, \text{ for any } \varepsilon > 0. \tag{5.10}$$

Let $\rho = 1/C_3$ and $\alpha \in (0,1)$ to be determined. It follows from (5.10), (5.2), and (5.9) that

$$\begin{aligned}(1-\varepsilon)\, e_{\ell+1} + \rho\eta_{\ell+1} &\le e_\ell + \rho\beta\eta_\ell + C_0\varepsilon^{-1}\hat{o}_\ell,\\ &\le \alpha(1-\varepsilon)e_\ell + \{[1-\alpha(1-\varepsilon)]C_1 + \rho\beta\}\,\eta_\ell + C_0\varepsilon^{-1}\hat{o}_\ell.\end{aligned}$$

Let α solve $\alpha\rho = [1-\alpha(1-\varepsilon)]C_1 + \rho\beta$. By requiring $\varepsilon < \rho(1-\beta)/C_1$, we obtain

$$\alpha = \frac{C_1 + \rho\beta}{(1-\varepsilon)C_1 + \rho} < 1.$$

The proof is complete. □

The next lemma deals with data oscillation reduction on two nested meshes.

Lemma 5.2. *Let $\mathcal{T}_h$ be a conforming refinement of $\mathcal{T}_H$ and $\mathcal{R}_H$ be the set of refined elements in $\mathcal{T}_H$. Then*

$$\operatorname{osc}^2_{\mathcal{T}_h}(f,\mathcal{T}_h) \le \operatorname{osc}^2_{\mathcal{T}_H}(f,\mathcal{T}_H) - \lambda_* \operatorname{osc}^2_{\mathcal{T}_H}(f,\mathcal{R}_H),$$

where $\lambda_* = 1 - 2^{-\frac{2}{n}}$.

Proof. Recall that $\omega_T := \{t \in \mathcal{T}_h : t \subset T\}$ for $T \in \mathcal{R}_H$. Then

$$\begin{aligned}\sum_{t\in\omega_T} h_t^2\|f - f_{\mathcal{T}_h}\|_t^2 &= \sum_{t\in\omega_T} |t|^{\frac{2}{n}}\|f - f_{\mathcal{T}_h}\|_t^2\\ &= 2^{-\frac{2}{n}} h_T^2 \sum_{t\in\omega_T}\|f - f_{\mathcal{T}_h}\|_t^2 \le 2^{-\frac{2}{n}} h_T^2\|f - f_{\mathcal{T}_H}\|_T^2,\end{aligned}$$

which implies

$$\sum_{t\subset T,\ T\in\mathcal{R}_H} \operatorname{osc}^2_{\mathcal{T}_h}(f,t) \le 2^{-\frac{2}{n}} \operatorname{osc}^2_{\mathcal{T}_H}(f,\mathcal{R}_H).$$

Therefore

$$\sum_{t\subset T,\ T\in\mathcal{R}_H} \operatorname{osc}^2_{\mathcal{T}_h}(f,T) + \lambda_* \operatorname{osc}^2_{\mathcal{T}_H}(f,\mathcal{R}_H) \le \operatorname{osc}^2_{\mathcal{T}_H}(f,\mathcal{R}_H). \tag{5.11}$$

For $T \in \mathcal{T}_H\backslash\mathcal{R}_H$,

$$\operatorname{osc}^2_{\mathcal{T}_h}(f,T) = \operatorname{osc}^2_{\mathcal{T}_H}(f,T). \tag{5.12}$$

Combining (5.11) and (5.12) completes the proof. □

The next theorem shows that AMFEM is a contraction.

Theorem 5.2 (contraction). *Let $\{\sigma_\ell, \mathcal{T}_\ell\}_{\ell\ge 0}$ be a sequence of solutions and meshes produced by AMFEM. For any $0 < \varepsilon < (1-\beta)/(C_1C_3)$, there exist $\rho, \zeta > 0$ and $0 < \gamma < 1$ depending only ε, θ and $\mathcal{T}_0$ such that,*

$$\begin{aligned}&(1-\varepsilon)\|\sigma - \sigma_{\ell+1}\|^2 + \rho\eta^2_{\mathcal{T}_{\ell+1}}(\sigma_{\ell+1},\mathcal{T}_{\ell+1}) + \zeta \operatorname{osc}^2_{\mathcal{T}_{\ell+1}}(f,\mathcal{T}_{\ell+1})\\ \le&\gamma\left\{(1-\varepsilon)\|\sigma - \sigma_\ell\|^2 + \rho\eta^2_{\mathcal{T}_\ell}(\sigma_\ell,\mathcal{T}_\ell) + \zeta \operatorname{osc}^2_{\mathcal{T}_\ell}(f,\mathcal{T}_\ell)\right\}.\end{aligned}$$

Proof. Let $F_\ell = (1-\varepsilon)\|\sigma-\sigma_\ell\|^2 + \rho\eta^2_{\mathcal{T}_\ell}(\sigma_\ell, \mathcal{T}_\ell)$. Theorem 5.1 and Lemma 5.2 read

$$F_{\ell+1} \le \alpha F_\ell + C_4 \hat{o}_\ell, \tag{5.13}$$

$$o_{\ell+1} \le o_\ell - \lambda_* \hat{o}_\ell, \quad 0 < \lambda_* < 1. \tag{5.14}$$

Let $\zeta = \lambda_*^{-1} C_4$. Combining (5.13), (5.14), and

$$\rho o_\ell \le \rho \eta_\ell \le F_\ell,$$

we have

$$\begin{aligned} F_{\ell+1} + \zeta o_{\ell+1} &\le \alpha F_\ell + \zeta o_\ell \\ &\le (\alpha + \zeta\alpha_1\rho^{-1})F_\ell + \zeta(1-\alpha_1)o_\ell \\ &= \gamma\left(F_\ell + \frac{\zeta(1-\alpha_1)}{\alpha+\zeta\alpha_1\rho^{-1}} o_\ell\right), \end{aligned} \tag{5.15}$$

where $\gamma := \alpha + \zeta\alpha_1\rho^{-1}$ and $\alpha_1 \in (0,1)$ is a constant to be determined. By requiring

$$\alpha + \alpha_1\zeta\rho^{-1} < 1, \quad \frac{1-\alpha_1}{\alpha+\zeta\alpha_1\rho^{-1}} \le 1, \tag{5.16}$$

(5.15) implies

$$F_{\ell+1} + \zeta o_{\ell+1} \le \gamma(F_\ell + \zeta o_\ell).$$

(5.16) can be satisfied by selecting

$$\frac{\rho(1-\alpha)}{\rho+\zeta} \le \alpha_1 < \min\left(1, \frac{\rho(1-\alpha)}{\zeta}\right).$$

The proof is complete. □

The methods used above to prove convergence have some similarities to prior work. Our treatment of oscillation, however, uses properties of $\hat{o}_\ell$ that create distinct implementation and efficiency improvements. To clarify this point, next we compare our convergence proof with [13] and [27].

In [13], oscillation is not included in the error indicator and therefore there is no control on o_ℓ in their quasi-orthogonality result (4.6). To enforce the strict reduction on $o_{\ell+1} \le \kappa o_\ell$ for some $\kappa < 1$, the AMFEM in [13] imposed a separate marking for data oscillation. Our convergence analysis shows that the marking for data oscillation is somehow artificial. The convergence of AMFEM can be achieved by a single marking step based on the estimator. This improvement essentially results from the sharper quasi-orthogonality Theorem 4.1 with the local data oscillation $\hat{o}_\ell$, which can be canceled using Lemma 5.2 on the oscillation reduction.

The second author [27] considered adaptive methods for the Hoge Laplacian problem (2.4) on the de Rham complex with index $1 \le k \le n$. Of particular interest here is the case $k = n$ for the mixed formulation of Poisson's equation. In particular, the AMFEM in [27] is a contraction in the error $\|\sigma-\sigma_h\|^2 + \hat{\zeta}\|d(\sigma-\sigma_h)\|^2 + \hat{\rho}\hat{\eta}^2_{\mathcal{T}_h}(\sigma_h, \mathcal{T}_h)$, which is generically of lower order than the total error in Theorem 5.2. Since [27] considered the error $\|\sigma-\sigma_h\|_{H\Lambda^{k-1}}$ in the V-norm instead of the L^2-norm, an elementary quasi-orthogonality (Lemma 4.1 in [27])

$$\|\sigma-\sigma_h\|^2 \le \frac{1}{1-\varepsilon}\|\sigma-\sigma_H\|^2 - \|\sigma_h-\sigma_H\|^2 + \frac{\varepsilon}{1-\varepsilon}\|f_{\mathcal{T}_h} - f_{\mathcal{T}_H}\|^2$$

is enough for convergence analysis there.

5.2. Optimality of AMFEM

The next theorem is devoted to a discrete upper bound, which is a common ingredient of optimal complexity proofs in the literature. Similar bound for the Hodge Laplacian problem has already been established in [27] by using Demlow's technique in [16]. Since the estimator $\eta_{\mathcal{T}_h}$ is different from the one when $k = n$ in [27], we sketch the proof here.

Theorem 5.3 (discrete upper bound). *Let $\mathcal{T}_h$ be a conforming refinement of $\mathcal{T}_H$ and $\mathcal{R}_H$ be the set of refined elements. There exists $\widetilde{\mathcal{R}}_H \supset \mathcal{R}_H$, which is the union of $\mathcal{R}_H$ and a collection of neighboring simplices of $\mathcal{R}_H$ with $\#\widetilde{\mathcal{R}}_H - \#\mathcal{R}_H \le C$, such that*

$$\|\sigma_h - \sigma_H\|^2 \le C_5 \eta^2_{\mathcal{T}_H}(\sigma_H, \widetilde{\mathcal{R}}_H).$$

Proof. The proof requires similar ingredients needed to prove the continuous upper bound. We first perform the discrete Hodge decomposition of $\sigma_h - \sigma_H$.

$$\begin{aligned}\sigma_h - \sigma_H &= P_{\mathfrak{B}_h}(\sigma_h - \sigma_H) + P_{\mathfrak{H}_h}(\sigma_h - \sigma_H) + P_{\mathfrak{Z}_h^{\perp}}(\sigma_h - \sigma_H)\\ &= (\sigma_h - P_{\mathfrak{Z}_h^{\perp}}\sigma_H) - P_{\mathfrak{B}_h}\sigma_H - P_{\mathfrak{H}_h}\sigma_H.\end{aligned}$$

Then each component can be estimated by the same procedure in the proof of continuous upper bound. With minimal modifications in the proofs of Lemmas 3.1, 3.2, and 3.3, we have

$$\|\sigma_h - P_{\mathfrak{Z}_h^{\perp}}\sigma_H\| \le C\,\mathrm{osc}_{\mathcal{T}_h}(f, \mathcal{R}_H), \tag{5.17a}$$

$$\|P_{\mathfrak{B}_h}\sigma_H\| = \langle -\sigma_H, d\varphi_h\rangle, \qquad \varphi_h \in V_h^{k-2}, \tag{5.17b}$$

$$\|P_{\mathfrak{H}_h}\sigma_H\| \le C_{\mathcal{T}_0}\|\sigma_h - \sigma_H\|, \quad C_{\mathcal{T}_0} < 1. \tag{5.17c}$$

To obtain the localized bound

$$\|P_{\mathfrak{B}_h}\sigma_H\| \le C\Big(\sum_{T\in\widetilde{\mathcal{R}}_H} h_T^2\|\delta\sigma_H\|^2 + h_T\|[\![\mathrm{tr}\star\sigma_H]\!]\|^2_{\partial T}\Big)^{\frac{1}{2}}, \tag{5.18}$$

we start from (5.17b) and using equations (4.11)-(4.17) in [16]. In the end, the discrete upper bound is proved by following the proof of Theorem 3.2 and using (5.17) and (5.18). ☐

Let $\mathbb{T}_N = \{\mathcal{T}$ is a conforming refinement of $\mathcal{T}_0 : \#\mathcal{T} - \#\mathcal{T}_0 \le N\}$. For $s > 0$, we define the approximation classes

$$\begin{aligned}\mathcal{A}_s &:= \{\tau \in H\Lambda^{n-1}(\Omega) : |\tau|_s := \sup_{N>0}\Big(N^s \inf_{\mathcal{T}\in\mathbb{T}_N}\inf_{\tau_{\mathcal{T}}\in V^n(\mathcal{T})}\|\tau - \tau_{\mathcal{T}}\|\Big) < \infty\},\\ \mathcal{A}_s^o &:= \{g \in L^2\Lambda^n(\Omega) : |g|_s^o := \sup_{N>0}\Big(N^s \inf_{\mathcal{T}\in\mathbb{T}_N}\mathrm{osc}_{\mathcal{T}}(g, \mathcal{T})\Big) < \infty\}.\end{aligned}$$

To prove the quasi-optimality, an extra module APPROX in [13] was assumed. Here we do not use APPROX. However, as in the classical AFEM literature [12,36], we make the following assumptions.

Assumption 5.1.

1. *The marking parameter $\theta \in (0, \theta_*)$, where $\theta_*^2 = \min(1, \frac{C_2}{C_5})$.*
2. *The marking step marks a subset $\mathcal{M}_\ell$ with minimal cardinality.*

3. *The accumulative cardinality of marked triangles satisfies* (5.1).

The threshold θ_* for marking parameter θ comes from the next lemma.

Lemma 5.3 (optimal marking). *Let $\mathcal{T}$ be a conforming refinement of $\mathcal{T}_0$ and $\sigma_{\mathcal{T}} \in V^n(\mathcal{T})$ be the solution of* (2.9) *on $\mathcal{T}$. Set $\mu = 1 - \frac{\theta^2}{\theta_*^2}$. Let $\mathcal{T}_*$ be a conforming refinement of $\mathcal{T}$, such that the finite element solution $\sigma_{\mathcal{T}_*} \in V^n(\mathcal{T}_*)$ satisfies*

$$\|\sigma - \sigma_{\mathcal{T}_*}\|^2 + \operatorname{osc}^2_{\mathcal{T}_*}(f, \mathcal{T}_*) \leq \mu \left\{\|\sigma - \sigma_{\mathcal{T}}\|^2 + \operatorname{osc}^2_{\mathcal{T}}(f, \mathcal{T})\right\}. \tag{5.19}$$

Then the set of enlarged refined elements $\widetilde{\mathcal{R}}$ in Theorem 5.3 *verifies the Dörfler marking property*

$$\eta_{\mathcal{T}}(\sigma_{\mathcal{T}}, \widetilde{\mathcal{R}}) \geq \theta \eta_{\mathcal{T}}(\sigma_{\mathcal{T}}, \mathcal{T}).$$

Proof. By Theorem 3.3 and (5.19),

$$\begin{aligned}(1-\mu)C_2\eta^2_{\mathcal{T}}(\sigma_{\mathcal{T}}, \mathcal{T}) &\leq (1-\mu)\big(\|\sigma - \sigma_{\mathcal{T}}\|^2 + \operatorname{osc}^2_{\mathcal{T}}(f, \mathcal{T})\big)\\ &\leq \|\sigma - \sigma_{\mathcal{T}}\|^2 - \|\sigma - \sigma_{\mathcal{T}_*}\|^2 + \operatorname{osc}^2_{\mathcal{T}}(f, \mathcal{T}) - \operatorname{osc}^2_{\mathcal{T}_*}(f, \mathcal{T}_*)\\ &\leq \|\sigma_{\mathcal{T}} - \sigma_{\mathcal{T}_*}\|^2.\end{aligned} \tag{5.20}$$

In the last step, we use the triangle inequality and $\operatorname{osc}_{\mathcal{T}_*}(f, \mathcal{T}_*) \leq \operatorname{osc}_{\mathcal{T}}(f, \mathcal{T})$. It then follows from (5.20) and Theorem 5.3 that

$$\eta^2_{\mathcal{T}}(\sigma_{\mathcal{T}}, \widetilde{\mathcal{R}}) \geq \frac{(1-\mu)C_2}{C_5}\eta^2_{\mathcal{T}}(\sigma_{\mathcal{T}}, \mathcal{T}).$$

The proof is complete by $\theta_*^2 \leq C_2/C_5$. □

Combining the optimal marking Lemma 5.3 ,the contraction Theorem 5.2, and the lower bound Theorem 3.3, the quasi-optimality of AMFEM follows from the same proof in [12], see Lemma 5.10 and Theorem 5.11 there for details.

Theorem 5.4 (quasi-optimality). *Assume that* (5.1) *holds. If $\sigma \in \mathcal{A}_s$ and $f \in \mathcal{A}_s^o$, then there exists C_6 depending only on θ, s, and $\mathcal{T}_0$, such that*

$$\left\{\|\sigma - \sigma_N\|^2 + \operatorname{osc}^2_{\mathcal{T}_N}(\sigma_N, \mathcal{T}_N)\right\}^{\frac{1}{2}} \leq C_6(\|\sigma\|_{\mathcal{A}_s} + \|f\|_{\mathcal{A}_s^o})(\#\mathcal{T}_N - \#\mathcal{T}_0)^{-s}.$$

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