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NUMERICAL ANALYSIS OF ELLIPTIC HEMIVARIATIONAL INEQUALITIES FOR SEMIPERMEABLE MEDIA*

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Abstract

In this paper, we consider elliptic hemivariational inequalities arising in applications in semipermeable media. In its general form, the model includes both interior and boundary semipermeability terms. Detailed study is given on the hemivariational inequality in the case of isotropic and homogeneous semipermeable media. Solution existence and uniqueness of the problem are explored. Convergence of the Galerkin method is shown under the basic solution regularity available from the existence result. An optimal order error estimate is derived for the linear finite element solution under suitable solution regularity assumptions. The results can be readily extended to the study of more general hemivariational inequalities for non-isotropic and heterogeneous semipermeable media with interior semipermeability and/or boundary semipermeability. Numerical examples are presented to show the performance of the finite element approximations; in particular, the theoretically predicted optimal first order convergence in H^1 norm of the linear element solutions is clearly observed.

Mathematics subject classification: 65N30, 49J40
Key words: Hemivariational inequality, Interior semipermeability, Boundary semipermeability, Finite element method, Error estimate.

1. Introduction

Variational inequalities for flow problems through porous media are studied in [9]. Such variational inequality problems adopt monotone semipermeability relations for the media. In [19], extension of the problems is made for semipermeable media to allow non-monotone semipermeability relations, leading to hemivariational inequalities. In both these references, semipermeability on the boundary or in the domain is considered.

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Since the pioneering work by Panagiotopoulos in early 1980s ([18]) on variational problems with nonconvex and generally nondifferentiable super-potentials, hemivariational inequalities have attracted steady attention from the research communities in mathematics, physical sciences and engineering. The formulation of hemivariational inequalities provides a useful framework to both theoretically and numerically treat application problems involving non-monotone, nonsmooth and multivalued constitutive laws, forces, and boundary conditions. Hemivariational inequalities have been shown very useful across a variety of subjects. Mathematical theory, numerical approximations and applications of hemivariational inequalities can be found in several monographs, e.g., [4, 15–17, 20]. The number of research papers on hemivariational inequalities is growing rapidly. The reference [15] discusses finite element approximations of hemivariational inequalities, including their convergence; however, no error estimates are provided. Recently, optimal order error estimates are derived for numerical solutions of hemivariational inequalities. The first paper along this direction is [12] where optimal order error estimates for the linear finite element solutions for some stationary hemivariational and variational-hemivariational inequalities are derived. This paper is followed by numerous papers on optimal order error estimates of the linear finite element solutions for various hemivariational inequalities of different form, e.g., [3] for the numerical solution of a hyperbolic hemivariational inequality, and [2] for the numerical solution of an evolutionary variational–hemivariational inequality. A general framework is presented on convergence analysis and error estimation for internal approximations of elliptic hemivariational inequalities in [13], and that for variational–hemivariational inequalities in [14]. In [11], a comprehensive convergence analysis and error estimation are given for both internal and external approximations of stationary variational–hemivariational inequalities and hemivariational inequalities. In all these references on numerical analysis of hemivariational inequalities, the application background is contact mechanics.

The purpose of this paper is to study and approximate elliptic hemivariational inequalities for the semipermeable media. The general hemivariational inequality incorporates both the interior and boundary semipermeability. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, i.e., Ω is an open, bounded and connected region in $\mathbb{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$. Here the positive integer d is the dimension of the problem under consideration. Since the boundary is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}$ is defined a.e. on $\partial\Omega$. We split the boundary $\partial\Omega$ into two non-overlapping and measurable parts Γ_0 and Γ_1 with $\mathrm{meas}\,(\Gamma_0) > 0$:

$$\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}. \tag{1.1}$$

We will specify a Dirichlet boundary condition on Γ_0 and a Neumann inclusion condition on Γ_1. The pointwise formulation of the model problem is as follows:

$$-\Delta u = f, \qquad \text{in } \Omega, \tag{1.2}$$

$$u = 0, \qquad \text{on } \Gamma_0, \tag{1.3}$$

$$-\frac{\partial u}{\partial \nu} \in \partial j_2(u), \qquad \text{on } \Gamma_1. \tag{1.4}$$

The differential equation (1.2) corresponds to the case of isotropic and homogeneous media (cf. [9,19]). Here, ∂j_2 is the generalized subdifferential of a locally Lipschitz continuous function j_2 (cf. Section 2). For simplicity, we let the Dirichlet boundary value to be zero in (1.3). The problem with a nonzero Dirichlet boundary value on Γ_0 can be handled with the standard technique (cf. e.g., [1, Subsection 8.4.2]). To allow the interior semipermeability condition, we