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STABILITY OF THE STOCHASTIC θ-METHOD FOR SUPER-LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY*

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Abstract

This paper deals with numerical stability properties of super-linear stochastic differential equations with unbounded delay. Sufficient conditions for mean square and almost sure decay stability of the above system and its stochastic θ-method approximation are investigated in this paper. The author establishes numerical stability under a monotone-type condition in unbounded delay setting. An example is presented to illustrate the result.

Mathematics subject classification: 60H10, 65C20, 65L20
Key words: Unbounded delay, Monotone condition, Polynomial condition, Stochastic θ-method, Decay stability.

1. Introduction

Stochastic differential equations (SDEs) with delay have attracted more and more interest in many different disciplines, in particular in biology [1]. But its analytic solutions are not widely available, which is the main motivation for the development of numerical methods. Indeed, numerical methods for SDEs with fixed delay have returned many results recently, such as convergence [2], stability [3], and dissipativity [4]. Many SDEs with time-depending delay and unbounded delay are not included in the above results. The following stochastic pantograph equations are some of the most interesting cases:

$$dx(t) = f(x(t), x(\alpha t), t)dt + g(x(t), x(\alpha t), t)dw(t), \qquad \alpha \in (0, 1), \quad t \geq 0.$$

It obtained results for the above system. Zhang et al. [5] obtained stability of numerical method for stochastic pantograph equations. A new predictor-corrector method for stochastic pantograph equations was given in [6].

Numerical stability plays an important role in numerical analysis. Many results were established under the linear growth condition [7–10] and the one-sided linear growth condition [11–14]. In the above literature, the diffusion coefficients of the equation were required to satisfy the linear growth condition. This excludes many important classes of stochastic systems, for example, the following well-known stochastic Lotka-Volterra model (see [15]):

$$\mathrm{d}x(t) = \mathrm{diag}(x_1, \cdots, x_n(t))[(b + Ax(t))\mathrm{d}t + x(t)\mathrm{d}w(t)],$$

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here b and A are constant. Some stability results for numerical approximations for super-linear diffusion were concluded. Mao and Szpruch [16] obtained asymptotic stability of the implicit numerical method in a polynomial growth setting for SDEs without delay. Several years later, Huang studied the mean-square stability of two classes of the theta method for SDEs with fixed delay under a coupled condition in [4]. Chen and Wu [17] investigated almost sure stability for SDEs with fixed delay under a monotone-type condition recently.

To the best of our knowledge, in a super-linear diffusion and unbounded delay setting, there is only one result as follows. In 2016, Zhou and Hu [18] established the stability of Backward Euler-Maruyama (BEM) approximation for stochastic pantograph differential equations under polynomial growth conditions. There are three significant differences between this paper and [18]. First stochastic θ-method will be the focus of this article, which is an extension of the BEM method. In fact, the stochastic θ-method will become the BEM method if $\theta = 1$. Second, this paper considers general unbounded delay, while in [18] the time-depending delay is still a fixed pantograph delay. Finally, we obtained stability under monotone conditions, which is more relaxed than under polynomial growth conditions.

Unless otherwise specified, the following notations were used throughout this paper. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If $a, b \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the largest integer number less than or equal to a, $\lceil a \rceil$ denotes the smallest integer number more than or equal to a, $a \vee b$ denotes the maximum of a and b, and $a \wedge b$ denotes the minimum of a and b. If A is a vector or matrix, A^T denotes the transpose of A, $|A| = \sqrt{\mathrm{trace}(A^T A)}$ denotes the trace norm of A. Let $\mathbb{R}_+ = [0, \infty)$ and $\tau_0 \in \mathbb{R}_+$ is a fixed constant. $\mathbf{C}([-\tau_0, 0]; \mathbb{R}^n)$ denotes the family of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$. Let $\mathbf{C}^b_{\mathcal{F}_0}([-\tau_0, 0]; \mathbb{R}^n)$ be the family of $\mathcal{F}_0$-measurable bounded $\mathbf{C}([-\tau_0, 0]; \mathbb{R}^n)$-valued random variables $\xi = \xi(t) : -\tau_0 \le t \le 0$. The inner product of $X, Y \in \mathbb{R}^n$ is denoted by $\langle X, Y \rangle$ or $X^T Y$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \ge 0}$ satisfying the usual conditions; that is, it is right continuous and increasing, while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $w(t)$ be a d-dimensional Brownian motion defined on this probability space. For notational simplicity, the author uses the convention that *const* represents a generic constant, which values may be different for different appearances.

The aim of this paper is to examine the decay stability of stochastic θ-method approximation for the following general unbounded delay SDEs with monotone conditions:

$$dx(t) = f(x(t), y(t), t)dt + g(x(t), y(t), t)dw(t), t \ge 0, \tag{1.1}$$

with initial data $\xi \in \mathbf{C}^b_{\mathcal{F}_0}([-\tau_0, 0]; \mathbb{R}^n)$, where $y(t) = x(t - \delta(t))$, $\delta(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $f(x, y, t)$: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$; $g(x, y, t)$: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n \times m}$ are Borel measurable.

We always assume $f(0, 0, t) \equiv 0$ and $g(0, 0, t) \equiv 0$ for the stability purpose of this paper. So Eq. (1.1) admits a trivial solution $x(t, 0) \equiv 0$. And assume that $\delta(0) \le \tau_0$ and

$$\eta \triangleq \inf_{t \ge 0} (t - \delta(t))' > 0. \tag{1.2}$$

These imply that $t - \delta(t) \ge -\tau_0$ and is a strictly monotonic increasing function for all $t \ge 0$. Condition (1.2) is a very relaxed restriction and the delay terms in this paper cover many general unbounded delays, for example $\delta(t) = \frac{t}{\pi} \arctan t$. But [18] considered the fixed pantograph delay which is a special kind of unbounded delay and many unbounded delays are not included.

As is done in [19], to consider the asymptotic stability with general decay rate, the following ψ-type function will be introduced, which will be used as the decay function.

Definition 1.1. *The function ψ: $R \to (0, \infty)$ is said to be a ψ-type function if ψ satisfies the following conditions:*