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ALTERNATING DIRECTION IMPLICIT SCHEMES FOR THE TWO-DIMENSIONAL TIME FRACTIONAL NONLINEAR SUPER-DIFFUSION EQUATIONS*

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Abstract

As is known, there exist numerous alternating direction implicit (ADI) schemes for the two-dimensional linear time fractional partial differential equations (PDEs). However, if the ADI schemes for linear problems combined with local linearization techniques are applied to solve nonlinear problems, the stability and convergence of the methods are often not clear. In this paper, two ADI schemes are developed for solving the two-dimensional time fractional nonlinear super-diffusion equations based on their equivalent partial integro-differential equations. In these two schemes, the standard second-order central difference approximation is used for the spatial discretization, and the classical first-order approximation is applied to discretize the Riemann-Liouville fractional integral in time. The solvability, unconditional stability and L_2 norm convergence of the proposed ADI schemes are proved rigorously. The convergence order of the schemes is $O(\tau + h_x^2 + h_y^2)$, where τ is the temporal mesh size, h_x and h_y are spatial mesh sizes in the x and y directions, respectively. Finally, numerical experiments are carried out to support the theoretical results and demonstrate the performances of two ADI schemes.

Mathematics subject classification: 65M06, 65M12, 35R11
Key words: Time fractional super-diffusion equation, Nonlinear system, ADI schemes, Stability, Convergence.

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1. Introduction

In this paper, we consider the following two-dimensional time fractional nonlinear super-diffusion ($1 < \alpha < 2$) problems

$$ {}_0^C D_t^\alpha u = \Delta u + f(x,y,t,u), \qquad (x,y) \in \Omega,\ 0 < t \le T, \tag{1.1} $$

with boundary condition

$$ u(x,y,t) = \psi(x,y,t), \qquad (x,y) \in \partial\Omega,\ 0 < t \le T, \tag{1.2} $$

and initial conditions

$$ u(x,y,0) = \phi(x,y), \qquad u_t(x,y,0) = \varphi(x,y), \qquad (x,y) \in \overline{\Omega}, \tag{1.3} $$

where Δ is the two-dimensional Laplacian, $\Omega = (0,L_x) \times (0,L_y)$, $\partial\Omega$ and $\overline{\Omega}$ are the boundary and the closure of Ω respectively, $f(x,y,t,u)$ is a nonlinear function of unknown u and fulfills a Lipschitz condition with respect to u, and $\psi(x,y,t)$, $\phi(x,y)$ and $\varphi(x,y)$ are known sufficiently smooth functions. ${}_0^C D_t^\alpha u$ denotes the Caputo derivative of order α defined as

$$ {}_0^C D_t^\alpha u(x,y,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{\partial^2 u(x,y,s)}{\partial s^2} ds. \tag{1.4} $$

When $\alpha = 1$ and $\alpha = 2$, Eq. (1.1) represents a diffusion equation and a wave equation, respectively. For $1 < \alpha < 2$, Eq. (1.1) is expected to interpolate the diffusion and the wave phenomena, thus in this case it could be referred to as the time fractional super-diffusion or diffusion-wave equations. Eq. (1.1) has been widely applied in the modeling of anomalous diffusive processes and the description of viscoelastic damping materials, etc. [3, 10, 13, 15, 16, 24, 27].

Similarly to the integer-order diffusion or wave differential equations, it's usually very d-ifficult to obtain the analytical solution of time fractional super-diffusion equations [2, 21, 23], especially for the nonlinear case. Thus there has been a growing interest to develop numerical approaches for solving time fractional super-diffusion equations.

In recent years, many published papers (see, e.g., [4,5,11,12,14,18,22,25,28,33]) are devoted to numerical methods for solving the one-dimensional time fractional diffusion equations. Sun and Wu in [28] constructed a classical $3 - \alpha$ order ($1 < \alpha < 2$) approximation for Caputo derivative, and then proposed a fully discrete difference scheme by introducing two new variables to transform the original equation into a low order system of equations, and gave the error analysis. In [18], Lin and Xu designed a finite difference/spectrral method based on a finite difference scheme in time and Legendre spectral methods in space. And stability and convergence of this method were rigourously established. Lv and Xu in [22] improved the results of [18] and obtained a higher order method for time fractional diffusion equations. Li et al. [14] used finite difference method in time direction and finite element method in space direction for time-space fractional diffusion-wave equations, and analyzed the semidiscrete and fully discrete numerical approximations. Huang et al. [11] constructed two finite difference schemes to solve a class of time fractional diffusion-wave equations based on their equivalent partial integro-differential equations, and proved that the proposed two schemes were convergent with the first-order accuracy in temporal direction and the second-order accuracy in spatial direction. In [25], Mustapha and Schötzau established the well-posedness of an *hp*-version time-stepping