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A DISCONTINUOUS GALERKIN METHOD BY PATCH RECONSTRUCTION FOR BIHARMONIC PROBLEM*

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Abstract

We propose a new discontinuous Galerkin method based on the least-squares patch reconstruction for the biharmonic problem. We prove the optimal error estimate of the proposed method. The two-dimensional and three-dimensional numerical examples are presented to confirm the accuracy and efficiency of the method with several boundary conditions and several types of polygon meshes and polyhedral meshes.

Mathematics subject classification: 49N45, 65N21.
Key words: Least-squares problem, Reconstructed basis function, Discontinuous Galerkin method, Biharmonic problem.

1. Introduction

The biharmonic boundary value problem is a fourth-order elliptic problem that models the thin plate bending problem in continuum mechanics, and describes slow flows of viscous incompressible fluids. Finite element methods have been employed to approximate this problem from its initial stage and by now there are many successful finite element methods for this problem [1].

Recently, the discontinuous Galerkin (DG) method [2–7] has been developed to solve the biharmonic problem. The DG method employs discontinuous basis functions that render great flexibility in the mesh partition and also provide a suitable framework for p-adaptivity. Higher order polynomials are easily implemented in DG method, which may efficiently capture the smooth solutions. To achieve higher accuracy, DG method requires a large number of degrees of freedom on a single element, which gives an extremely large linear system [8, 9]. As a

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compromise of the standard FEM and the DG method, Brenner et al [10, 11] developed the so-called C^0 interior penalty Galerkin method (C^0 IPG). This method applies standard continuous C^0 Lagrange finite elements to the interior penalty Galerkin variational formulation, which admits the optimal error estimate with less local degree of freedoms compared with standard DG method.

The aim of this work is to apply the patch reconstruction finite element method proposed in [12] to the biharmonic problem. The main idea of the proposed method is to construct a piecewise polynomial approximation space by patch reconstruction. The approximation space is discontinuous across the element face and has only one degree of freedom located inside each element, which is a sub-space of the commonly used discontinuous Galerkin finite element space. One advantage of the proposed method is the number of the unknown is maintained under a given mesh partition with the increasing of the order of accuracy. Moreover, the reconstruction procedure can be carried out over any mesh such as an arbitrary polygonal mesh. Given such reconstructed approximation space, we solve the biharmonic problem in the framework of interior penalty discontinuous Galerkin (IPDG). Based on the approximation properties of the approximation space established in [13] and [12], we analyze the proposed method in the framework of IPDG. The performance of the proposed method is verified by a series of numerical tests with different complexity, which is comparable with the C^0 IPG method while the basis functions in our method should be pre-computed, nevertheless, such basis functions may be reused.

The article is organized as follows. In Section 2, we demonstrate the reconstruction procedure of the approximation space and develop the corresponding approximation properties of such a space. Next, the IPDG method with such a reconstructed approximation space is proposed and analyzed in Section 3. In Section 4, we test the proposed method by several two-dimensional and three-dimensional biharmonic boundary value problems, which include smooth solution as well as nonsmooth solution for various boundary conditions and different types of meshes.

Throughout this paper, the constant C is a generic constant that may change from line to line, but is independent of the mesh size h.

2. Reconstruction Operator

Let $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ be a bounded convex domain. Let $\mathcal{T}_h$ be a collection of Ne polygonal elements that partition Ω. We denote all interior faces of $\mathcal{T}_h$ as $\mathcal{E}_h^i$ and the set of the boundary faces as $\mathcal{E}_h^b$, and then $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$ is then a collection of all $(d-1)$-dimensional faces of all elements in $\mathcal{T}_h$. Let $h_K = \mathrm{diam} K$ and $h = \max_{K \in \mathcal{T}_h} h_K$. We assume that $\mathcal{T}_h$ satisfies the shape-regular conditions used in [14, 15], which read: there exist

- an integer number N independent of h;
- a real positive number σ independent of h;
- a compatible sub-decomposition $\widetilde{\mathcal{T}_h}$ into shape-regular triangles or quadrilaterals, or mix of both triangles and quadrilaterals,

such that

(A1) any polygon $K \in \mathcal{T}_h$ admits a decomposition $\widetilde{\mathcal{T}_h}$ formed by less than N triangles;