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ERROR ANALYSIS OF A STABILIZED FINITE ELEMENT METHOD FOR THE GENERALIZED STOKES PROBLEM*

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Abstract

This paper is devoted to the establishment of sharper *a priori* stability and error estimates of a stabilized finite element method proposed by Barrenechea and Valentin [3] for solving the generalized Stokes problem, which involves a viscosity ν and a reaction constant σ. With the establishment of sharper stability estimates and the help of *ad hoc* finite element projections, we can explicitly establish the dependence of error bounds of velocity and pressure on the viscosity ν, the reaction constant σ, and the mesh size h. Our analysis reveals that the viscosity ν and the reaction constant σ respectively act in the numerator position and the denominator position in the error estimates of velocity and pressure in standard norms without any weights. Consequently, the stabilization method is indeed suitable for the generalized Stokes problem with a small viscosity ν and a large reaction constant σ. The sharper error estimates agree very well with the numerical results.

Mathematics subject classification: 65N12, 65N15, 65N30, 76M10.
Key words: Generalized Stokes equations, Stabilized finite element method, Error estimates.

1. Introduction and Preliminaries

Let Ω be an open bounded polygonal domain in $\mathbb{R}^d$ ($d = 2$ or 3) with boundary $\partial\Omega$. In this paper, we will study the stabilized $\mathcal{C}^0$ finite element approximations, proposed by Barrenechea and Valentin in [3], to the following system of generalized Stokes equations with the homogeneous velocity boundary condition:

$$\begin{cases} \sigma \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^d$ is the velocity field and $p : \overline{\Omega} \to \mathbb{R}$ is the pressure; $\nu > 0$ is the viscosity constant; $\sigma > 0$ is the reaction constant; and $\boldsymbol{f} : \Omega \to \mathbb{R}^d$ is a given source-like function in $(L^2(\Omega))^d$.

In general, the finite element approach for solving problem (1.1) is posed as a velocity-pressure mixed formulation in the standard Galerkin method. However, it is well known that, for stable and optimally accurate approximations, the pair $(\boldsymbol{V}_h, Q_h)$ of finite element spaces for the mixed formulation must satisfy the so-called *inf-sup condition,*

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \geq c\|q\|_0 \qquad \forall\ q \in Q_h, \tag{1.2}$$

see, e.g., [8], [10], and [25]. This condition prevents the use of standard equal order $\mathcal{C}^0$ finite element spaces for velocity and pressure with respect to the same triangulation that are the most attractive from the viewpoint of implementation. In order to circumvent the inf-sup condition, a class of so-called stabilized finite element methods (FEMs) has been developed and intensively studied for more than thirty years, see, e.g., [6, 7, 9, 11, 12, 15, 21, 22, 26, 27, 32, 33, 35, 39]. The stabilized FEMs are formed by adding to the discrete mixed formulation of the generalized Stokes problem (1.1) with some consistent variational terms, relating to the residuals of the partial differential equations (cf. [14, 16, 19, 20, 23, 30, 31, 36, 37]). With suitable stabilization parameters, the stabilized FEMs are successful in circumventing the above inf-sup condition.

Typically, the generalized Stokes problem (1.1) may arise from the time discretization (cf. [38]) of transient Stokes equations or full Navier-Stokes equations by means of an operator splitting technique, where the reaction constant is given by $\sigma = c(\delta t)^{-1}$ and δt is the time step. For problems involving fast chemical reactions, a small time step, namely a large σ, is needed in order to account for the stiffness due to the fast reaction. However, in the context of stabilization methods, it has been observed that the pressure instabilities may be caused as the time step δt becomes small compared to the spatial grid size h. Therefore, in recent years, there has been increasingly a great deal of attention on the theoretical and computational studies of small time-step instabilities when implicit, finite difference time integration is applied in combination with finite element stabilization in the spatial semi-discretization, see, e.g., [4, 5, 14, 17, 19, 31]. Nowadays, it has been extensively recognized that the stabilized FEMs are most effective in dealing with the instability in the finite element solution.

In [3], Barrenechea and Valentin proposed a stabilized FEM for solving the generalized Stokes problem (1.1) in 2D. The unusual feature of this stabilized FEM is that it involves the subtraction of the stabilization term $\sum_{K \in \mathcal{T}_h} \tau_K(\sigma \boldsymbol{u}_h, \sigma \boldsymbol{v})_{0,K}$ from the original discrete mixed finite element formulation. Numerical results provided in [3] show that the proposed method can achieve high accuracy and stability. More remarkably, it has been numerically verified in [3] that for a fixed small viscosity ν, the H^1 errors of the resulting finite element solutions of velocity appear to be uniform in the reaction coefficient σ when σ is large enough.

In this paper, with the help of analysis of the finite element projections for velocity and pressure, together with a trick using a function $\xi(\cdot)$ of the ratio between ν and σh_K^2, we are able to derive sharper error estimates for the Barrenechea-Valentin stabilized $\mathcal{C}^0$ FEM that will be briefly stated at the end of this section. We first establish two sharper stability estimates, and then establish the explicit dependence of the error bounds on the viscosity ν, the reaction constant σ, and the mesh size h. The significant new findings in our analysis can be summarized as follows. The analysis reveals that the viscosity constant ν and the reaction constant σ respectively act in the numerator position and the denominator position in the error estimates