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UNCONDITIONAL SUPERCONVERGENCE ANALYSIS OF AN H^1-GALERKIN MIXED FINITE ELEMENT METHOD FOR TWO-DIMENSIONAL GINZBURG-LANDAU EQUATION*

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Abstract

An H^1-Galerkin mixed finite element method (MFEM) is discussed for the two-dimensional Ginzburg-Landau equation with the bilinear element and zero order Raviart-Thomas element ($Q_{11} + Q_{10} \times Q_{01}$). A linearized Crank-Nicolson fully-discrete scheme is developed and a time-discrete system is introduced to split the error into two parts which are called the temporal error and the spatial error, respectively. On one hand, the regularity of the time-discrete system is deduced through the temporal error estimation. On the other hand, the superconvergent estimates of u in H^1-norm and $\vec{q}$ in $H(div;\Omega)$-norm with order $O(h^2 + \tau^2)$ are obtained unconditionally based on the achievement of the spatial result. At last, a numerical experiment is included to illustrate the feasibility of the proposed method. Here, h is the subdivision parameter and τ is the time step.

Mathematics subject classification: 65N15, 65N30.
Key words: The two-dimensional Ginzburg-Landau equation, H^1-Galerkin MFEM, Temporal and spatial errors, Unconditionally, Superconvergent results.

1. Introduction

Consider the following Ginzburg-Landau equation:

$$\begin{cases} u_t - (a_1 + ib_1)\Delta u + (a_2 + ib_2)|u|^2 u - \gamma u = 0, & (X,t) \in \Omega \times (0,T], \\ u = 0, & (X,t) \in \partial\Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^2$ is a rectangle with the boundary $\partial\Omega$, $0 < T < \infty$, u is an unknown complex function, $a_1 > 0, a_2 > 0, b_1, b_2$ are given real constants, u_0 is a given complex function, γ is the coefficient of the linear evolution term, and $\gamma > 0$.

The Ginzburg-Landau equation is an important nonlinear evolution equation and has various forms. (1.1) is a kind of Ginsburg-Landau equations discussed by numerical methods deeply. For example, semi-discrete and the implicit Euler fully-discrete approximations were studied in [1]. [2] considered the convergence behavior of some finite difference schemes. Avoiding the difficulty in estimating the numerical solutions, [2] proved that all the schemes were of the second-order convergence in L^2-norm by an induction argument. In [3], a linearized Crank-Nicolson-Galerkin method was proposed to solve the nonlinear and coupled partial differential equations and some error estimates were deduced. In [4], three difference schemes of the Ginzburg-Landau equation

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in two dimensions were presented, in which the nonlinear term was discretized such that the nonlinear iteration was not needed in computation. The plane wave solution of the equation was studied and the truncation errors of the three schemes were obtained. The stability of the two difference schemes was proved by induction method and the time-splitting method was analyzed by linearized analysis. The algebraic multi-grid method was used to solve the three large linear systems of the schemes.

However, the above results were all derived under some time-step restrictions such as $\tau = O(h^{\frac{d}{2}}), (d = 2, 3)$ in [1,3], $\tau = o(h^2)$ in [2] and $\tau = O(h)$ in [4]. In order to avoid such defects, lots of literature have been devoted to such topic. For example, the so-called splitting technique was proposed in [5] and developed in the articles [6-15] where a corresponding time-discrete system was constructed to split the error into two parts, the temporal error and the spatial error. Based on novel special technique, [16] derived the unconditional superclose results for Sobolev equation with conforming mixed FEM.

Different from [6-16] the main aim of this paper is to study the unconditional superconvergence analysis for (1.1) with an H^1-Galerkin MFEM. As we all know that the two spaces used in the traditional MFEMs must be chosen carefully so that they satisfy an LBB stability condition. Fortunately, [17] proposed an H^1-Galerkin MFEM in which the above LBB condition is not required and thus it has been applied widely to many partial differential equations, such as the regularized long wave equation [18], hyperbolic equations [19-21], evolution equation with a positive-type Memory Term [22], integro-differential equations [23-24], parabolic equations [25-26], Sobolev equation [27], and so on.

In the present work, motivated by [6-15], we aim to the superclose estimates of u in H^1-norm and $\vec{p} = \nabla u$ in $H(div; \Omega)$-norm unconditionally through choosing the element pair $Q_{11}/Q_{10} \times Q_{01}$. On one hand, a time-discrete system with the solutions U^n and $\vec{P}^n$ is introduced and the boundedness of $\bar{\partial}_t U^n$ in H^2-norm and $\bar{\partial}_t \vec{P}^n$ in H^1-norm are given. On the other hand, we split U_h^n and $\vec{P}_h^n$, which are the numerical solutions, into several parts to avoid the appearance of them in L^∞-norms or stronger norms directly. To confirm our theoretical analysis, an numerical example is carried out. It is worthy to mention that, for the special characteristics of the nonlinear Ginzburg-Landau equation, we employ a new trick taking difference between two time levels n and $n-1$ of the temporal error equation to obtain the stricter regularity of the solutions about the time-discrete system.

Throughout this paper, we denote the natural inner product in $L^2(\Omega)$ by $(\cdot, \cdot)$ and the norm by $\|\cdot\|_0$, and let $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. Further, we use the classical Sobolev spaces $W^{m,p}(\Omega), 1 \le p \le \infty$, denoted by $W^{m,p}$, with norm $\|\cdot\|_{m,p}$. When $p = 2$, we simply write $\|\cdot\|_{m,p}$ as $\|\cdot\|_m$. Besides, we define the space $L^p(a, b; Y)$ with the norm $\|f\|_{L^p(a,b;Y)} = (\int_a^b \|f(\cdot, t)\|_Y^p dt)^{\frac{1}{p}}$, and if $p = \infty$, the integral is replaced by the essential supremum.

2. A Linearized Galerkin Approximation Scheme

Let Ω be a rectangle in (x, y) plane with edges parallel to the coordinate axes, Γ_h be a polygon domain in (x, y) plane with edges parallel to the coordinate axes, Γ_h be a regular rectangular subdivision of Ω. For given $K \in \Gamma_h$, let the four vertices and edges are $a_i, i = 1 \sim 4$ and $l_i = \overline{a_i a_{i+1}}, i = 1 \sim 4 \pmod 4$, respectively.