Journal of Computational Mathematics
Vol.37, No.4, 2019, 458–474.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1804-m2017-0240

INTERIOR ESTIMATES OF SEMIDISCRETE FINITE ELEMENT METHODS FOR PARABOLIC PROBLEMS WITH DISTRIBUTIONAL DATA*

Li Guo
School of Data and Computer Science, Sun Yat-Sen University, Guangzhou, China
Email: guoli7@mail.sysu.edu.cn
Hengguang Li
Department of Mathematics, Wayne State University, Michigan, USA
Email: li@wayne.edu
Yang Yang
Department of Mathematical Sciences, Michigan Technological University, Michigan, USA
Email: yyang7@mtu.edu

Abstract

Let $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, be a bounded d-polytope. Consider the parabolic equation on Ω with the Dirac delta function on the right hand side. We study the well-posedness, regularity, and the interior error estimate of semidiscrete finite element approximations of the equation. In particular, we derive that the interior error is bounded by the best local approximation error, the negative norms of the error, and the negative norms of the time derivative of the error. This result implies different convergence rates for the numerical solution in different interior regions, especially when the region is close to the singular point. Numerical test results are reported to support the theoretical prediction.

Mathematics subject classification: 65M15, 65M60
Key words: Parabolic problems, Distributional data, Finite element methods, Interior estimates, Well-posedness, Singularity.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, be a bounded d-polytope and $\Omega_T := \Omega \times (0, T]$. Namely, Ω is a line segment for $d = 1$, a polygon for $d = 2$, and a polyhedron for $d = 3$. Denote by $\delta_z(x) = \delta(x - z)$ the Dirac delta function at $z \in \Omega$. We consider a parabolic problem with the homogeneous Dirichlet boundary condition

$$u_t - \Delta u = f, \qquad \text{in} \quad \Omega_T, \tag{1.1a}$$

$$u = 0, \qquad \text{on} \quad \partial\Omega \times [0, T], \tag{1.1b}$$

$$u(\cdot, 0) = u_0, \qquad \text{on} \quad \Omega \times \{t = 0\}, \tag{1.1c}$$

where $u_0 \in L^2(\Omega)$ and $f = g\delta_z$ for $g \in L^2(0, T; C(\bar{\Omega}))$. The finite element approximations for parabolic equations with sufficiently smooth solutions have been well investigated in the literature (see, e.g., [3, 21, 29]). The study of numerical methods for parabolic equations with less regular data has become increasingly popular in recent years. We refer to [5, 14, 15] for equations with singular solutions due to the non-smoothness in the domain and in the coefficient

* Received October 5, 2017 / Revised version received February 5, 2018 / Accepted April 12, 2018 / Published online September 10, 2018 /

of the differential operator. Some recent results on point-wise approximations can be found in [12, 13] for fully discrete methods (finite element method in space and discontinuous Galerkin method in time). For numerical analysis on parabolic problems with Dirac delta functions, we mention [8, 25] and references therein. In these works, the global convergence of the numerical scheme on the entire domain was obtained approximating the singular solution.

Partial differential equations with the δ-function sources have many applications in astrophysics and oil reservoir simulations. Especially in the latter case, an interesting model is the two-phase flow displacement in porous medium which can be described by a parabolic system. Moreover, the injection and production wells can be represented by point sources and sinks, respectively, which can be approximated by δ-singularities with different strengths. In such problems, the exact solutions are not smooth and high-order numerical schemes can yield poor convergence or strong oscillations in a vicinity (pollution region) of the singularity. In [30], Yang and Shu applied discontinuous Galerkin methods to solve linear hyperbolic equations with δ-function source terms in one space dimension and the size of the pollution region was proved to be of order $\mathcal{O}(h^{\frac{1}{2}})$, where h is the mesh size.

Note that the distributional data in Eq. (1.1) can lead to singular solutions, for which the global approximation may not be of high-order accuracy even when high-order finite element methods are used. Meanwhile, the numerical approximation in certain interior regions is often more interesting in practice. In this paper, we study the interior error estimate of the semidiscrete finite element method for Eq. (1.1). In particular, we first derive the well-posedness of the weak solution for Eq. (1.1) in suitable Sobolev spaces (Theorem 2.1). This result extends the well-posedness result in [8, 16] on convex domains to general polytopal domains. Then, we show that away from the singular point z, the solution possesses higher interior regularity (Corollary 2.1), which justifies the use of the L^2 and H^1 norms of the error in such interior regions for our error analysis. The main result regarding the interior error estimate is summarized in Theorem 4.1, in which we obtain that the L^2 and H^1 norms of the error in an interior region away from z are determined by three components: the best local approximation error from the finite element space, the negative norms of the interior error, and the negative norms of the time derivative of the local error. Namely, the interior convergence may be of higher order compared with the global convergence, which is affected by the regularity of u and u_t. Applying this result to regions close to z, we further formulate an interior estimate (Corollary 4.1) that depends on the distance to the singular point. This implies that as the region gets closer to z, the interior convergence rate can slow down and eventually resemble the global convergence rate.

For elliptic boundary value problems, the finite element interior estimates have been studied in a series of papers [20, 22–24]. These results show that the error in an interior region is bounded by the best local approximation error and the error in negative norms. Thus, the interior error estimates in this paper extend these results to parabolic problems by including additional effects from the time derivative of the solution. We also mention that for parabolic equations, an interior finite element analysis was derived in [27] using the energy method on uniform meshes with specific conditions. In this paper, we use a more direct method to obtain the interior error analysis, especially when distributional data is present. In addition, our results apply to general quasi-uniform meshes.

The rest of the paper is organized as follows. In Section 2, we introduce the notation and derive the well-posedness and regularity results for the parabolic problem (1.1). In Section 3, we formulate the semidiscrete finite element approximation and recall useful properties of the numerical scheme. In Section 4, we obtain the interior error estimates for the parabolic