J. Math. Study
doi: 10.4208/jms.v52n1.19.04

Vol. **52**, No. 1, pp. 38-52
March 2019

On Multivariate Fractional Taylor's and Cauchy' Mean Value Theorem

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Received September 29, 2017; Accepted January 8, 2018

Abstract. In this paper, a generalized multivariate fractional Taylor's and Cauchy's mean value theorem of the kind

$$f(x,y)=\sum_{j=0}^{n}\frac{D^{j\alpha}f(x_0,y_0)}{\Gamma(j\alpha+1)}+R_n^\alpha(\xi,\eta),\qquad \frac{f(x,y)-\sum\limits_{j=0}^{n}\frac{D^{j\alpha}f(x_0,y_0)}{\Gamma(j\alpha+1)}}{g(x,y)-\sum\limits_{j=0}^{n}\frac{D^{j\alpha}g(x_0,y_0)}{\Gamma(j\alpha+1)}}=\frac{R_n^\alpha(\xi,\eta)}{T_n^\alpha(\xi,\eta)},$$

where $0<\alpha\leq 1$, is established. Such expression is precisely the classical Taylor's and Cauchy's mean value theorem in the particular case $\alpha=1$. In addition, detailed expressions for $R_n^\alpha(\xi,\eta)$ and $T_n^\alpha(\xi,\eta)$ involving the sequential Caputo fractional derivative are also given.

AMS subject classifications: 65M70, 65L60, 41A10, 60H35

Key words: Sequential Caputo fractional derivative, generalized Taylor's mean value theorem, generalized Taylor's formula, generalized Cauchy' mean value theorem, generalized Cauchy's formula

1 Introduction

The ordinary Taylor's formula has been generalized by many authors. Riemann [1] had already written a formal version of the generalized Taylor's series:

$$f(x+h)=\sum_{m=-\infty}^{\infty}\frac{h^{m+r}}{\Gamma(m+r+1)}(D_a^{-(m+r)}f)(x),\tag{1.1}$$

where $D_a^{-(m+r)}$ is the Riemann-Liouville fractional integral of order $m+r$.

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Afterwards, Watanable [2] obtained the following relation:

$$f(x)=\sum_{k=-m}^{n-1}\frac{(x-x_0)^{\alpha+k}}{\Gamma(\alpha+k+1)}(D_a^{m+r}f)(x_0)+R_{n,m}, \tag{1.2}$$

with $m<\alpha, a\le x_0<x$, and

$$R_{n,m}=(D_{x_0}^{-(\alpha+n)}D_a^{\alpha+n}f)(x)+\frac{1}{\Gamma(-\alpha-m)}\int_a^{x_0}(x-t)^{-\alpha-m-1}(D_a^{\alpha-m-1}f)(t)dt,$$

where $D_a^{\alpha+n}$ is the Riemann-Liouville fractional derivative of order $\alpha+n$.

On the other hand, a variant of the generalized Taylor's series was given by Dzherbashyan and Nersesyan [3]. For f having all of the required continuous derivatives, they obtained

$$f(x)=\sum_{k=0}^{m-1}\frac{(D^{(\alpha_k)}f)(0)}{\Gamma(1+\alpha_k)}x^{\alpha_k}+\frac{1}{\Gamma(1+\alpha_m)}\int_0^x(x-t)^{\alpha_m-1}(D^{(\alpha_m)}f)(t)dt, \tag{1.3}$$

where $0<x,\alpha_0,\alpha_1,...,\alpha_m$ is an increasing sequence of real numbers such that $0<\alpha_k-\alpha_{k-1}\le 1, k=1,...,m$ and $D^{(\alpha_m)}f=I_0^{1-(\alpha_k-\alpha_{k-1})}D_0^{1+\alpha_{k-1}}f$.

Under certain conditions for f and $\alpha\in[0,1]$, Trujillo *et al.* [4] introduce the following generalized Taylor's mean value theorem:

$$\begin{aligned}
&f(x)=\sum_{j=0}^{n}\frac{c_j(x-a)^{(j+1)\alpha-1}}{\Gamma((J+1)\alpha)}+R_n(f;\xi),\\
&R_n(f;\xi)=\frac{(D_a^{(n+1)\alpha}f)(\xi)}{\Gamma((n+1)\alpha+1)}\cdot(x-a)^{(n+1)\alpha}, \qquad a\le\xi\le x,\\
&c_j=\Gamma(a)[(x-a)^{1-\alpha}D_a^{j\alpha}f)(a+), \qquad j=0,1,...,n
\end{aligned} \tag{1.4}$$

and the sequential fractional Riemann-Liouville derivative is denoted by

$$D_a^{n\alpha}=D_a^{\alpha}\cdot D_a^{\alpha}\cdot...\cdot D_a^{\alpha}(n-times).$$

Recently, Odibat and Shawagfeh [5] obtain a new generalized Taylor's mean value theorem of this kind

$$f(x)=\sum_{j=0}^{n}\frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)}(D_a^{j\alpha}f)(a)+\frac{(D_a^{(n+1)\alpha}f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha} \tag{1.5}$$

with $a\le\xi\le x$, where $D^{j\alpha}$ is the sequential fractional Caputo derivative.

In 2005, Pecaric *et al.* [6] deduced the Cauchy type mean value theorem for the sequence fractional Riemann-Liouville derivative from known mean value theorem of the Lagrange type.