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Rigidity of Minimizers in Nonlocal Phase Transitions II

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Abstract. In this paper we extend the results of [12] to the borderline case $s=\frac{1}{2}$. We obtain the classification of global bounded solutions with asymptotically flat level sets for semilinear nonlocal equations of the type

$$\Delta^{\frac{1}{2}}u=W'(u) \quad \text{in } \mathbb{R}^n,$$

where W is a double well potential.

Key Words: De Giorgi's conjecture, fractional Laplacian.

AMS Subject Classifications: 35J61

1 Introduction

We continue the study initiated in [12] for the classification of global bounded solutions with asymptotically flat level sets for nonlocal semilinear equations of the type

$$\Delta^s u=W'(u) \quad \text{in } \mathbb{R}^n,$$

where W is a double well potential.

The case $s\in(\frac{1}{2},1)$ was treated in [12] while $s\in(0,\frac{1}{2})$ was considered by Dipierro, Serra and Valdinoci in [5]. In this paper we obtain the classification of global minimizers with asymptotically flat level sets in the remaining borderline case $s=\frac{1}{2}$. All these works were motivated by the study of semilinear equations for the case of the classical Laplacian $s=1$, and their connection with the theory of minimal surfaces, see [2,4,9,10]. It turns out that when $s\in[\frac{1}{2},1)$, the rescaled level sets of u still converge to a minimal surface while for $s\in(0,\frac{1}{2})$ they converge to an s-nonlocal minimal surface, see [13].

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We consider the Ginzburg-Landau energy functional with nonlocal interactions corresponding to $\Delta^{1/2}$,

$$J(u,\Omega)=\frac{1}{4}\int_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\complement\Omega\times\complement\Omega)}\frac{(u(x)-u(y))^2}{|x-y|^{n+1}}dxdy+\int_\Omega W(u)dx,$$

with $|u|\le 1$, and W a double-well potential with minima at 1 and -1 satisfying

$$\begin{aligned}&W\in C^2([-1,1]),\quad W(-1)=W(1)=0,\quad W>0\quad \text{on}\quad (-1,1),\\ &W'(-1)=W'(1)=0,\quad W''(-1)>0,\quad W''(1)>0.\end{aligned}$$

Critical functions for the energy J satisfy the Euler-Lagrange equation

$$\Delta^{1/2}u=W'(u),$$

where $\Delta^{1/2}u$ is defined as

$$\Delta^{1/2}u(x)=PV\int_{\mathbb{R}^n}\frac{u(y)-u(x)}{|y-x|^{n+1}}dy.$$

Our main result provides the classification of minimizers with asymptotically flat level sets.

Theorem 1.1. Let u be a global minimizer of J in $\mathbb{R}^n$. If the 0 level set $\{u=0\}$ is asymptotically flat at ∞, then u is one-dimensional.

The hypothesis that $\{u=0\}$ is asymptotically flat means that there exist sequences of positive numbers θ_k, l_k and unit vectors ξ_k with $l_k\to\infty$, $\theta_k l_k^{-1}\to 0$ such that

$$\{u=0\}\cap B_{l_k}\subset\{|x\cdot\xi_k|<\theta_k\}.$$

By saying that u is one-dimensional we understand that u depends only on one direction ξ, i.e., $u=g(x\cdot\xi)$.

As in [12], we obtain several corollaries. We state two of them.

Theorem 1.2. A global minimizer of J is one-dimensional in dimension $n\le 7$.

Theorem 1.3. Let $u\in C^2(\mathbb{R}^n)$ be a solution of

$$\Delta^{1/2}u=W'(u), \tag{1.1}$$

such that

$$|u|\le 1,\quad \partial_n u>0,\quad \lim_{x_n\to\pm\infty}u(x',x_n)=\pm 1. \tag{1.2}$$

Then u is one-dimensional if $n\le 8$.