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A Priori Error Estimate of Splitting Positive Definite Mixed Finite Element Method for Parabolic Optimal Control Problems

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Abstract. In this paper, we propose a splitting positive definite mixed finite element method for the approximation of convex optimal control problems governed by linear parabolic equations, where the primal state variable y and its flux σ are approximated simultaneously. By using the first order necessary and sufficient optimality conditions for the optimization problem, we derive another pair of adjoint state variables z and ω, and also a variational inequality for the control variable u is derived. As we can see the two resulting systems for the unknown state variable y and its flux σ are splitting, and both symmetric and positive definite. Besides, the corresponding adjoint states z and ω are also decoupled, and they both lead to symmetric and positive definite linear systems. We give some a priori error estimates for the discretization of the states, adjoint states and control, where Ladyzhenkaya-Babuska-Brezzi consistency condition is not necessary for the approximation of the state variable y and its flux σ. Finally, numerical experiments are given to show the efficiency and reliability of the splitting positive definite mixed finite element method.

AMS subject classifications: 49K20, 65M60

Key words: Parabolic optimal control, splitting mixed finite element method, positive definite, a priori error estimates, numerical experiments.

1. Introduction

Optimal control or design is crucial to many engineering applications. Efficient numerical methods are essential to successful applications of optimal control. Nowadays, the finite element method seems to be the most widely used numerical method in computing optimal control problems. There are indeed very extensive studies in the finite element approximation of various optimal control problems; see, for example,

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Refs. [1–17] for the a priori and a posteriori error analysis. Systematic introductions of the finite element method for PDEs and optimal control problems can be found in, for example, Refs. [18, 19, 21–24].

In many optimal control problems, the objective functional contains not only the state variable, but also its flux. For example, in the flow control problem, the flux stands for Darcy velocity and it is an important physics variable; or, in the temperature control problem, large temperature gradients during cooling or heating may lead to its destruction. Therefore, in these cases people pay their special attention on the flux of the primal state variable, see, Refs. [4, 6, 9, 11]. In this paper, we propose a splitting positive definite procedure to convex optimization problems governed by linear parabolic equations. In summary, the proposed splitting mixed finite element method has three typical advantages compared to the classical mixed finite element methods: First, it is not subjected to the Ladyzhenkaya-Babuska-Brezzi (LBB) consistency condition, so the choice of approximation function spaces becomes flexible. Second, it results in two symmetric and positive definite systems which can be solved using the popular preconditioned conjugate gradient (PCG) or algebraic multi-grid (AMG) solvers. Third, the resulting linear systems for the unknown state variable y and its flux σ, the corresponding adjoint states z and ω are decoupled, respectively, thus the computation load is greatly reduced.

In the rest, we consider the following linear-quadratic optimal control problem for the state variable y and the control variable u involving pointwise control constraints:

$$\min \int_0^T \left(g_1(y) + g_2(-\mathcal{A}\nabla y) + j(u)\right) dt, \tag{1.1}$$

subject to

$$\begin{cases} y_t - \mathrm{div}(\mathcal{A}\nabla y) = f + u, & \text{in } \ \Omega \times (0, T], \\ y = 0, & \text{on } \ \partial\Omega \times (0, T], \\ y(0) = y_0, & \text{in } \ \Omega, \end{cases} \tag{1.2}$$

and

$$u_a \leq u(x,t) \leq u_b, \ a.e. \text{ in } \ \Omega \times [0, T], \tag{1.3}$$

where $y_t = \frac{\partial y}{\partial t}$, $g_i(\cdot)$ $(i = 1, 2)$ and $j(\cdot)$ are given convex functionals. A precise formulation of this problem including a functional analytic setting is given in the next section.

The remainder of this paper is organized as follows. In Sect. 2, we first give a precise description of the optimal control problem and then derive the continuous optimality conditions. In Sect. 3, we construct a semi-discrete numerical scheme using the splitting mixed finite element approximation, and deduce the corresponding discrete optimality conditions. In Sect. 4, we derive a priori error estimates for the semi-discrete scheme with the control constrained by bilateral pointwise inequalities. In Sect. 5, we conduct some numerical experiments to observe the theoretical results of the numerical scheme.