Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2015.m1411

Vol. **9**, No. 1, pp. 1-50
February 2016

Numerical Approaches for Linear Left-invariant Diffusions on $SE(2)$, their Comparison to Exact Solutions, and their Applications in Retinal Imaging

Jiong Zhang [1,*], Remco Duits[1,2,*], Gonzalo Sanguinetti[2]
and Bart M. ter Haar Romeny[3,1]

[1] *Department of Biomedical Engineering, Biomedical Image Analysis (BMIA)*
[2] *Department of Mathematics and Computer Science, Image Science and Technology (IST/e)*
[1,2] *Eindhoven University of Technology, P.O. Box 513, 5600 MB, Eindhoven, The Netherlands.*
[3] *Northeastern University, Shenyang, China.*

Received 13 April 2014; Accepted (in revised version) 26 January 2015

Abstract. Left-invariant PDE-evolutions on the roto-translation group $SE(2)$ (and their resolvent equations) have been widely studied in the fields of cortical modeling and image analysis. They include hypo-elliptic diffusion (for contour enhancement) proposed by Citti & Sarti, and Petitot, and they include the direction process (for contour completion) proposed by Mumford. This paper presents a thorough study and comparison of the many numerical approaches, which, remarkably, are missing in the literature. Existing numerical approaches can be classified into 3 categories: Finite difference methods, Fourier based methods (equivalent to $SE(2)$-Fourier methods), and stochastic methods (Monte Carlo simulations). There are also 3 types of exact solutions to the PDE-evolutions that were derived explicitly (in the spatial Fourier domain) in previous works by Duits and van Almsick in 2005. Here we provide an overview of these 3 types of exact solutions and explain how they relate to each of the 3 numerical approaches. We compute relative errors of all numerical approaches to the exact solutions, and the Fourier based methods show us the best performance with smallest relative errors. We also provide an improvement of Mathematica algorithms for evaluating Mathieu-functions, crucial in implementations of the exact solutions. Furthermore, we include an asymptotical analysis of the singularities within the kernels and we propose a probabilistic extension of underlying stochastic processes that overcomes the singular behavior in the origin of time-integrated kernels. Finally, we show retinal imaging applications of combining left-invariant PDE-evolutions with invertible orientation scores.

---

[*] Corresponding and joint main authors. *Email addresses:* `J.Zhang1@tue.nl` (J. Zhang), `R.Duits@tue.nl` (R. Duits), `g.r.sanguinetti@tue.nl` (G. Sanguinetti), `B.M.terhaarRomeny@tue.nl` (B.-t.-H. Romeny)

PACS: 35R03, 65M80, 65M06, 65M12, 62H35, 60H30

Key words: Brownian motion, Euclidean motion group, PDE's on $SE(2)$, Mathieu operators, contour completion, contour enhancement, retinal imaging.

---

1. Introduction

Hubel and Wiesel [38] discovered that certain visual cells in cats' striate cortex have a directional preference. It has turned out that there exists an intriguing and extremely precise spatial and directional organization into so-called cortical hypercolumns, see Fig. 1. A hypercolumn can be interpreted as a "visual pixel", representing the optical world at a single location, neatly decomposed into a complete set of orientations. Moreover, correlated horizontal connections run parallel to the cortical surface and link columns across the spatial visual field with a shared orientation preference, allowing cells to combine visual information from spatially separated receptive fields. Synaptic physiological studies of these horizontal pathways in cats' striate cortex show that neurons with aligned receptive field sites excite each other [15]. Apparently, the visual system not only constructs a score of local orientations, but also accounts for context and alignment by excitation and inhibition *a priori*, which can be modeled by left-invariant PDE's and ODE's on $SE(2)$ [6,8,11,12,17,19,21,24–26,29,32,43,46,47,50,51,58]. Motivated by the orientation-selective cells, so-called orientation scores are constructed by lifting all elongated structures (in 2D images) along an extra orientation dimension [21,24,41]. The main advantage of using the orientation score is that we can disentangle the elongated structures involved in a crossing allowing for a crossing preserving flow.

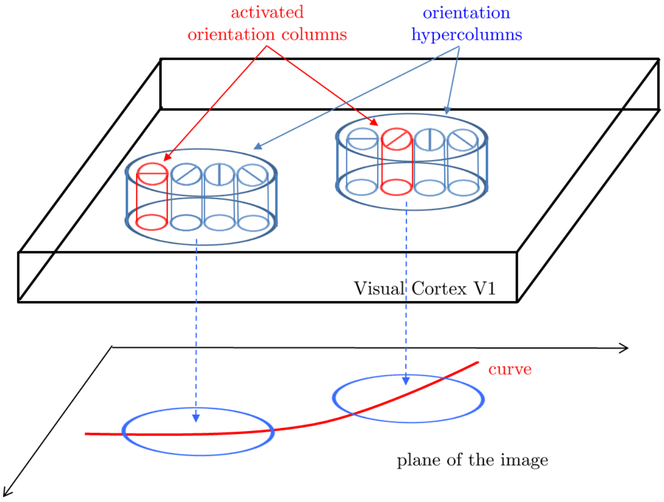

Figure 1: The orientation columns in the primary visual cortex.