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A Conservative Formulation and a Numerical Algorithm for the Double-Gyre Nonlinear Shallow-Water Model

Dongyang Kuang and Long Lee*

Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA.

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Abstract. We present a conservative formulation and a numerical algorithm for the reduced-gravity shallow-water equations on a beta plane, subjected to a constant wind forcing that leads to the formation of double-gyre circulation in a closed ocean basin. The novelty of the paper is that we reformulate the governing equations into a nonlinear hyperbolic conservation law plus source terms. A second-order fractional-step algorithm is used to solve the reformulated equations. In the first step of the fractional-step algorithm, we solve the homogeneous hyperbolic shallow-water equations by the wave-propagation finite volume method. The resulting intermediate solution is then used as the initial condition for the initial-boundary value problem in the second step. As a result, the proposed method is not sensitive to the choice of viscosity and gives high-resolution results for coarse grids, as long as the Rossby deformation radius is resolved. We discuss the boundary conditions in each step, when no-slip boundary conditions are imposed to the problem. We validate the algorithm by a periodic flow on an f-plane with exact solutions. The order-of-accuracy for the proposed algorithm is tested numerically. We illustrate a quasi-steady-state solution of the double-gyre model via the height anomaly and the contour of stream function for the formation of double-gyre circulation in a closed basin. Our calculations are highly consistent with the results reported in the literature. Finally, we present an application, in which the double-gyre model is coupled with the advection equation for modeling transport of a pollutant in a closed ocean basin.

AMS subject classifications: 65M08; 76B15
Key words: Double-gyre, reduced-gravity shallow-water equations, wave-propagation algorithm, fractional-step algorithm.

1. Introduction

The two-dimensional shallow-water equations govern the fluid motion in a thin layer. They can be used as a rational approximation to the three-dimensional Euler

*Corresponding author. *Email address:* dkuang@uwyo.edu (D.-Y. Kuang), llee@uwyo.edu (L. Lee)

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equations, with the assumption of hydrostaticity and shallow water depth (compared with the horizontal length scale). When wind forcing and latitude-dependent Coriolis forces are included, these equations represent a simple model for describing the depth-average dynamics of the oceans. Furthermore, if we include a Laplacian diffusion in the equations and impose Dirichlet boundary conditions on the velocity field, in particular the no-slip conditions, the equations are often used to simulate a mid-latitude closed ocean basin. In this paper we focus on a reduced-gravity shallow-water model formulated for studying the behavior of western boundary currents (WBCs) in mid latitudes [3]. In this ocean model water is assumed to consist of two layers of fluid, a single active layer of fluid of constant density ρ and variable thickness $h(x,y,t)$, overlying a deep and motionless layer of density $\rho+\Delta\rho$. Consequently, the motion of the upper layer represents the gravest baroclinic mode [3]. The model equations in nonconservative form are

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}+\frac{\partial (vh)}{\partial y}=0, \tag{1.1a}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-g_r\frac{\partial h}{\partial x}+(f_0+\beta y)v+\nu\nabla^2 u+F^u, \tag{1.1b}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-g_r\frac{\partial h}{\partial y}-(f_0+\beta y)u+\nu\nabla^2 v+F^v, \tag{1.1c}$$

where (u,v) is the velocity filed, h is the height field, $g_r=(\Delta\rho/\rho)g$ is the reduced gravity, and g is the acceleration of gravity. (F^u,F^v) is the external forcing term, such as the wind forcing [3,4,9,10]. With the imposition of no-slip boundary conditions on the velocity field (the height field is allowed to assume any value on the boundaries), equations (1.1a) describe a wind-driven, closed basin on a β plane. The equations are normally referred to as the double-gyre, wind-driven shallow-water model. This model is a convenient test bed for studying mid-latitude ocean dynamics [3,4,10].

The numerical algorithm MPDATA (Multidimensional Positive Definite Advection Transport Algorithm) has long been used to solve geophysical flows, such as flow governed by Eq. (1.1a). MPDATA is a two-pass scheme that preserves positive definite scalar transport functions with small oscillations [14–16]. Technically, the method belongs to the same class of non-oscillatory Lax-Wendroff algorithms such as FCT [20], TVD [17], and ENO [2]. Nevertheless, MPDATA was primarily developed for meteorological applications. The method focuses on reducing the implicit viscosity of the donor cell scheme, while retaining the virtues of positivity, low phase error, and simplicity of upstream differencing. However, the disadvantage of MPDATA is that the basic MPDATA is too diffusive, and enhanced MPDATA is too expensive [16]. We compare a basic MPDATA implementation described in [11] with the proposed algorithm for the double-gyre model in Section 3. For a thorough review of MPDATA, we refer the readers to [16].

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