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A Multilevel Method for the Solution of Time Dependent Optimal Transport

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Abstract. In this paper we present a new computationally efficient numerical scheme for the minimizing flow for the computation of the optimal L_2 mass transport mapping using the fluid approach. We review the method and discuss its numerical properties. We then derive a new scaleable, efficient discretization and a solution technique for the problem and show that the problem is equivalent to a mixed form formulation of a nonlinear fluid flow in porous media. We demonstrate the effectiveness of our approach using a number of numerical experiments.

AMS subject classifications: 15A12, 65F10, 65F15

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1. Introduction

Optimal mass transport is of cardinal importance in geoscience and engineering with other applications in econometrics, fluid dynamics, automatic control, transportation, statistical physics, shape optimization, expert systems, and meteorology [26, 31]. The problem was first formulated by the civil engineer Gaspar Monge in 1781, and concerns with finding an optimal way, in the sense of minimal transportation cost, of moving a pile of soil from one site to another. Much later the problem was extensively analyzed by Kantorovich [21], and is now known as the Monge-Kantorovich problem.

There are several formulations of the problem [2, 26, 31] of varying degrees of generality. Here we start with the formulation of the Monge-Kantorovich problem for smooth densities and domains in Euclidean space (for more general measures, see [2]). Let Ω_0 and Ω_1 be two diffeomorphic connected subdomains of $\mathbb{R}^d$, and let μ_0, μ_1 be

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Borel measures on Ω_0 and Ω_1, each with a strictly positive density function $\mu_0(x) \geq \mu_{\text{low}}^0 > 0$ and $\mu_1 \geq \mu_{\text{low}}^1 > 0$, respectively. Assume

$$\int_{\Omega_0} \mu_0(x)dx = \int_{\Omega_1} \mu_1(x)dx,$$

so that the same total mass is associated with Ω_0 and Ω_1.

Under some mild assumptions, the Monge-Kantorovich problem may be expressed as the following optimization problem

$$\min \quad M(u) := \frac{1}{p}\int_{\Omega} \mu_0(x)|u(x)|^p\, dx \tag{1.1a}$$

$$\text{s.t.} \quad c(u) = \det(I_d + \nabla u)\mu_1(x + u(x)) - \mu_0(x) = 0, \tag{1.1b}$$

where u is a $C^{1,\alpha}$ diffeomorphism from $\Omega_0 \to \Omega_1$. The constraint $c(u) = 0$ (the Jacobian equation) is often referred to as the mass preserving (MP) property. Here, we consider the classical case of $p = 2$ as well as $1 < p \leq 2$ and attempt to numerically address the limiting (ill-posed) case when $p = 1$.

Even with a simple, quadratic distance function, the problem (1.1) is regarded as a highly nonlinear equality constrained optimization problem. Extensive analysis as for the existence, uniqueness, and properties of the solution is available (see for example [2, 15, 31] and the references therein). However, while a large body of literature deals with the analysis of the problem, surprisingly a relatively small number of papers concern with finding numerical solutions to the problem, and even a smaller number of publications that deal with devising *efficient*, that is, scalable, numerical solutions for this challenging problem [3, 5, 11–13, 19, 24].

Generally speaking, numerical methods for the solution of the problem can be divided into three approaches. In the first approach, for the case $p = 2$, one utilizes the property that $u = \nabla\phi$ where ϕ is a concave function and solves the Monge-Ampère equation [14, 25]. The second approach attempts to tackle the constrained optimization problem head-on. Among this work is our previous algorithm [19].

A third approach for the solution of the problem, which is the starting point of this study was proposed in the seminal paper of Benamou and Brenier [5]. Their research reconstructs an optimal path from μ_0 to μ_1 by solving a convex optimization problem with a linear space-time transport partial differential equation as a constraint. Their approach is particularly useful if the transportation path is needed. Its disadvantage is that it increases the dimensionality of the problem by recasting the problem as a space-time control problem. In the original work of Benamou and Brenier, simple nodal discretization was used, combined with the augmented Lagrangian method for the solution of the problem. When reproducing the results of the paper we have observed some stability issues as well as deterioration of the algorithm for large-scale problems. As we show in this work this instability can be explained by analyzing their discretization using multigrid tools (Fourier analysis) that show that the discretization is not h-elliptic.