Numer. Math. Theor. Meth. Appl. Vol. **7**, No. 3, pp. 356-373
doi: 10.4208/nmtma.2014.1320nm August 2014

Superconvergence of a Galerkin FEM for Higher-Order Elements in Convection-Diffusion Problems

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Received 27 June 2013; Accepted (in revised version) 5 February 2014

Available online 12 August 2014

Abstract. In this paper we present a first supercloseness analysis for higher-order Galerkin FEM applied to a singularly perturbed convection-diffusion problem. Using a solution decomposition and a special representation of our finite element space, we are able to prove a supercloseness property of $p+1/4$ in the energy norm where the polynomial order $p \geq 3$ is odd.

AMS subject classifications: 65N12, 65N30, 65N50

Key words: Singular perturbation, layer-adapted meshes, superconvergence, postprocessing.

1. Introduction

Consider the convection dominated convection-diffusion problem

$$-\varepsilon\Delta u - (b\cdot\nabla)u + cu = f, \quad \text{in } \Omega = (0,1)^2, \tag{1.1a}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $c \in L_\infty(\Omega)$, $b \in W^1_\infty(\Omega)$, $f \in L_2(\Omega)$ and $0 < \varepsilon \ll 1$, assuming

$$c + \frac{1}{2}\operatorname{div} b \geq \gamma > 0. \tag{1.2}$$

For a problem with exponential layers, i.e. in the case $b_1(x,y) \geq \beta_1 > 0$, $b_2(x,y) \geq \beta_2 > 0$, we have for linear or bilinear elements in the so called energy norm

$$|||v|||_\varepsilon^2 := \varepsilon\|\nabla v\|_0^2 + \|v\|_0^2,$$

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where $\|\cdot\|_0$ denotes the usual L_2-norm, on a Shishkin mesh (for the exact definition see Section 2)

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \lesssim N^{-1} \ln N.$$

We use the notation $a \lesssim b$, if a generic constant C independent of ε and N exists with $a \leq Cb$.

However, for bilinear elements Zhang [23] and Linß [13] observed a supercloseness property: the difference between the Galerkin solution u^N and the standard piecewise bilinear interpolant u^I of the exact solution u satisfies

$$\left|\left|\left|u^I - u^N\right|\right|\right|_\varepsilon \lesssim (N^{-1} \ln N)^2.$$

Supercloseness is a very important property. It allows optimal error estimates in L_2 (Nitsche's trick cannot be applied), improved error estimates in L_∞ inside the layer regions and recovery procedures for the gradient, important in a posteriori error estimation.

In the last ten years supercloseness for bilinear elements was also proved for problems with characteristic layers [6], for S-type meshes [13], for Bakhvalov meshes [15] and for several stabilisation methods, including streamline diffusion FEM (SDFEM), continuous interior penalty FEM (CIPFEM), local projection stabilisation FEM (LPS-FEM) and discontinuous Galerkin (see e.g. [3, 7–9, 17, 18, 21]). Recently, even corner singularities were included in the analysis [14].

For $\mathcal{Q}_p$-elements with $p \geq 2$ the situation is very different. Using the so-called vertex-edge-cell interpolant πu [11, 12] instead of the standard Lagrange-interpolant with equidistant interpolation points, Stynes and Tobiska [19] proved for SDFEM (but not for the Galerkin FEM)

$$\left|\left|\left|\pi u - \tilde{u}^N\right|\right|\right|_\varepsilon \lesssim N^{-(p+1/2)},$$

where $\tilde{u}^N$ denotes the SDFEM solution. It is not clear whether this estimate is optimal. The numerical results of [4, 5] indicate for the Galerkin FEM and $p \geq 3$ a supercloseness property of order $p+1$ for two different interpolation operators. One of them is the vertex-edge-cell interpolator πu, the other one is the Gauss-Lobatto interpolation operator $I^N u$. For SDFEM, the order $p+1$ is observed numerically for all $p \geq 2$.

In the present paper we study the Galerkin FEM for odd p. We shall prove some supercloseness properties, but the achieved order is probably not optimal.

The paper is organised as follows. In Section 2 we provide descriptions of the underlying mesh, the numerical method and a solution decomposition. The main part is Section 3 where the proof of our assertion can be found. As the proof is rather technical we provide it in full only for $p = 3$ and demonstrate its generalisation for arbitrary odd $p \geq 5$. In Section 4 we present some numerical simulations.